Multiple Choice Questions 1 : Solutions

1. (i) **False** : Take $E = E^\circ = ]0, \frac{1}{2}[$ and $F = [0, 1]$. We have that $\frac{1}{2} \in \partial E$ but $\frac{1}{2} \notin \partial (E \cup F) = \partial [0, 1] = \{0, 1\}$.

(ii) **True** : If $x \in \partial (E \cup F)$, then by definition

$$\forall r > 0, B(x, r) \cap (E \cup F)^c \neq \emptyset \quad \text{and} \quad B(x, r) \cap (E \cup F) \neq \emptyset.$$ 

Therefore either for $E$ or $F$ (let say $E$), we have

$$\forall r > 0, B(x, r) \cap (E \cup F)^c \neq \emptyset \quad \text{and} \quad B(x, r) \cap E \neq \emptyset.$$ 

Since $E^c \supset (E \cup F)^c$,

$$\forall r > 0, B(x, r) \cap E^c \neq \emptyset \quad \text{and} \quad B(x, r) \cap E \neq \emptyset,$$

we finally get that $x \in \partial E$, and thus that $\partial (E \cup F) \subset \partial E \cup \partial F$. Conversely, let $x \in \partial E \cup F$. Therefore either $x \in \partial E$ or $x \in \partial F$, let say without loss of generality that $x \in \partial E$. We have that $\partial E \subset \overline{E}$ and $\overline{E} \cap F = \emptyset$, therefore $x \in \overline{E}^c$. Since $F$ is closed, $\overline{F}^c$ is open and thus,

$$\exists r_0 > 0 : \forall r \leq r_0, B(x, r) \subset B(x, r_0) \subset \overline{F}^c.$$ 

Therefore, for all $r \leq r_0$,

$$B(x, r) \cap E \neq \emptyset, \quad B(x, r) \cap E^c \neq \emptyset, \quad B(x, r) \cap F = \emptyset \quad \text{and} \quad B(x, r) \subset F^c.$$ 

Therefore, for all $r \leq r_0$,

$$B(x, r) \cap E \neq \emptyset \quad \text{and} \quad B(x, r) \cap (E \cup F)^c \neq \emptyset,$$

and thus

$$B(x, r) \cap (E \cup F) \neq \emptyset \quad \text{and} \quad B(x, r) \cap (E \cup F)^c \neq \emptyset,$$

what prove that $x \in \partial (E \cup F)$.

(iii) **False** : Let $E = ]0, 1[$ and $F = [1, 2]$. We have that $1 \in \partial E$, but

$$\partial (E \cup F) = \partial [0, 1] = \{0, 2\},$$

while

$$E^\circ \cap F^\circ = ]0, 1[ \cap ]1, 2[ = \emptyset.$$ 

(iv) **False** : same counter-example as (iii).
2. \( \mathcal{N} \) is not a norm since
\[
\mathcal{N}(2x) = \frac{\|2x\|_2}{1 + \|2x\|_2} = \frac{2\|x\|_2}{1 + 2\|x\|_2} < 2 \cdot \frac{\|x\|_2}{1 + \|x\|_2} = 2\mathcal{N}(x),
\]
and thus the homogeneity is not valid. On the other hand, \( d \) is indeed a distance. the fact that \( d(x, y) > 0 \) for all \( x \neq y \) and \( d(x, x) = 0 \) is clear. Let show the triangle inequality. First, let observe that
\[
t \mapsto \frac{t}{1 + t}
\]
is increasing on \([0, \infty)\). Also, we see that for \( u, v \geq 0 \),
\[
\frac{u}{1+u} + \frac{v}{1+v} - \frac{u+v}{1+u+v} = \frac{u(1+v)}{(1+u)(1+v)} + \frac{v(1+u)}{(1+v)(1+u)} - \frac{u+v}{1+u+v}
\]
\[= \frac{u+v+2uv}{1+u+v+uv} - \frac{u+v}{1+u+v}
\]
\[= \frac{uv(u+v+2)}{(1+u+v)^2 + (1+u+v)uv} \geq 0,
\]
and thus
\[
\frac{u+v}{1+u+v} \leq \frac{u}{1+u} + \frac{v}{1+v}, \tag{1}
\]
for all \( u, v \geq 0 \). Using the fact that \( \| \cdot \|_2 \) is a norm, we get that
\[
\|x-z\|_2 \leq \|x-y\|_2 + \|y-z\|_2,
\]
and thus, if \( x, y, z \in \mathbb{R} \),
\[
d(x, z) = \mathcal{N}(x-z) = \frac{\|x-z\|_2}{1 + \|x-z\|_2} \leq \frac{\|x-y\|_2 + \|y-z\|_2}{1 + \|x-y\|_2 + \|y-z\|_2}
\]
\[\leq \frac{\|x-y\|_2}{1 + \|x-y\|_2} + \frac{\|y-z\|_2}{1 + \|y-z\|_2} = \mathcal{N}(x-y) + \mathcal{N}(y-z) = d(x,y) + d(y,z),
\]
where the first inequality come from the fact that \( t \mapsto \frac{t}{1+t} \) is increasing. The claim follow.

3. (i) False : \( E \) is not closed. Indeed, let consider the sequence defined by \( x_n = (\frac{1}{n}, \frac{1}{n^{3/2}}) \). Then \( (x_n) \) is a sequence of \( E \). Moreover we have that
\[
\lim_{n \to \infty} x_n = (0, 0) \notin E,
\]
and thus \( E \) is not closed.
(ii) False : since $E$ is not closed.

(iii) True : Let show that $E^c$ is closed. Remark that

$$E^c = \{(x, y) \mid y \leq x^2 \text{ or } y \geq x\} \cup (-\infty, 0] \times \mathbb{R} \cup \{1, +\infty \times \mathbb{R}\}.$$ 

Since $\mathbb{R}$ and $[1, +\infty \times \mathbb{R}$ are closed, we just have to show that

$$\{(x, y) \mid x \in [0, 1] \text{ and } (y \leq x^2 \text{ or } y \geq x)\}$$ 

is closed (because a finite union of closed is closed). Let $((x_n, y_n))_{n \in \mathbb{N}}$ a convergent sequence of $E$. Let denote $(a, b)$ the limit. If the limit is $(0, 0)$ or $(1, 1)$ then we are done. So we suppose $(a, b) \notin \{(0, 0), (1, 1)\}$. Let

$$A = \{n \mid y_n \leq x_n^2\} := \{n_k\}_{k \in \mathbb{N}} \text{ and } B = \{n \mid y_n \geq x_n\} := \{m_k\}_{k \in \mathbb{N}}.$$ 

We necessarily have that $|A|$ or $|B|$ is finite. Indeed, if both where non-finite, then

$$b = \lim_{k \to \infty} y_{n_k} \leq \lim_{k \to \infty} x_{n_k}^2 = a^2$$ 

and

$$b = \lim_{k \to \infty} y_{m_k} \geq \lim_{k \to \infty} x_{n_k} = a,$$

and thus $a \leq b \leq a^2$. Since $a \in [0, 1]$ we have that $a^2 \leq a$ and thus we get $a = a^2 = b$ what implies that $(a, b) = (0, 0)$ or $(a, b) = (1, 1)$ which is impossible by our assumption on $(a, b)$. Suppose without loss of generality that $A$ is finite and $B$ is non-finite. In particular, there is $k \in \mathbb{N}$ such that $n \in B$ for all $n \geq k$. Therefore, $x_n \leq y_n$ for all $n \geq k$ and thus

$$a = \lim_{n \to \infty} x_n \leq \lim_{n \to \infty} y_n = b.$$ 

In particular $(a, b) \in E$, and thus $E$ is closed. Let show that

$$\partial E = \{(x, x) \mid x \in [0, 1]\} \cup \{(x, x^2) \mid x \in [0, 1]\}.$$ 

The fact that $\partial E \subset \{(x, x) \mid x \in [0, 1]\} \cup \{(x, x^2) \mid x \in [0, 1]\}$ is obvious, and thus, we just have to show the converse inclusion. Also, the fact that $(0, 0)$ and $(1, 1)$ are in $\partial E$ is obvious. Let $x \in ]0, 1[$ and consider the ball $B\left((x, x), \frac{1}{n}\right)$ with $\frac{1}{n}$ small enough to have $x - \frac{1}{n} > x^2$. In particular if $m > n$, we have that $(x, x - \frac{1}{m}) \in E$ and $(x, x + \frac{1}{m}) \in E^c$. Therefore

$$\forall n \in \mathbb{N}, B\left((x, x), \frac{1}{n}\right) \cap E \neq \emptyset \text{ and } B\left((x, x), \frac{1}{n}\right) \cap E^c \neq \emptyset.$$ 

Therefore $(x, x) \in \partial E$. Let $x \in ]0, 1[$, we consider $n$ big enough to have $x^2 + \frac{1}{n} < x$. Take $m > n$ and show that $(x, x^2 + \frac{1}{m}) \in E$ and $(x, x^2 - \frac{1}{m}) \in E^c$ what will prove the claim (to see things better, make a draw!).

4. (i) False : Take $E = [0, 1]$.

(ii) False : Take $E = [0, 1]$. 

3
(iii) **True**: Let show that \( E^c \) is open. Suppose by contradiction that it’s not open. So there is an \( x \in E^c \) such that for all \( n \),
\[
B\left(x, \frac{1}{n}\right) \cap E \neq \emptyset.
\]
So let \( x_n \in B(x, \frac{1}{n}) \cap E \) for all \( n \). In particular, \((x_n)\) is a sequence of \( E \) that converge to \( x \), and by assumption it converge in \( E \). Therefore \( x \in E \) which is a contradiction with \( x \in E^c \). Therefore \( E^c \) is open, and thus \( E \) is closed.

(iv) **False**: Take \( E = \mathbb{R}^n \).

5. (i) **False**: Since \( E \) is countable, it can’t contain any ball. Therefore it’s not open.

(ii) **False**: Same reason as previously.

(iii) **False**: Since \( \lim_{n \to \infty} \left( \frac{1}{n}, e^{-\left(e^{\log(n)}\right)^{3/2}} \right) = (0, 0) \notin E \), it can’t be closed.

(iv) **True**.