1. a. The solution is: \( y(t) = \int_0^t f(x)dx + y_0 \) \( \forall t \in I \).

2. b. Recall the counter-example from class: \( y' = 2\sqrt{|y|}, y_0 = 0 \) has 2 solutions on \([-1,1]\), which are \( y_1(t) = 0 \) and \( y_2(t) = t^2 \).

3. b. \( f_2 \) continuous and \( f_0 \) Lipschitz.

An equation of order 3 in dimension 1 = An equation of order 1 in dimension 3. So we consider \( \begin{bmatrix} y \\ y' \\ y'' \end{bmatrix} \) instead of \( y \).

Let \( Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \) and the equation becomes:

\[
Y' = \begin{bmatrix} y_2 \\ y_3 \\ y_3.f_2(x) + f_0(y_1) \end{bmatrix} := F(Y)
\]

And \( Y(0) = \begin{bmatrix} y_0 \\ y'_0 \\ y''_0 \end{bmatrix} \) On a bounded interval \( I \). If \( F \) is Lipschitz, then the problem admits a unique solution. Let \( Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \) and \( \tilde{Y} = \begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \\ \tilde{y}_3 \end{bmatrix} \) and we compute:

\[
\| F(Y) - F(\tilde{Y}) \|^2 \leq (y_2 - \tilde{y}_2)^2 + (y_3 - \tilde{y}_3)^2 + 2(f_2(x))^2(y_1 - \tilde{y}_2)^2 + 2(f_0(y_1) - f_0(\tilde{y}_1))^2
\]

(Here we used the fact that \((a + b)^2 \leq 2a^2 + 2b^2\).)

- If \( f_0 \) is Lipschitz, \( \exists L_0 \geq 0 \) a constant such that \( \forall y, \tilde{y}, (f_0(y) - f_0(\tilde{y}))^2 \leq L_0(y - \tilde{y})^2 \).
  Then we get:
  \[
  \| F(Y) - F(\tilde{Y}) \|^2 \leq 2L_0(y - \tilde{y})^2 + (1 + 2(f_2(x))^2)((y_2 - \tilde{y}_2)^2 + (y_3 - \tilde{y}_3)^2) \leq \max(2L_0, 1 + \max((f_2(x))^2, 1)) \cdot \| Y - \tilde{Y} \|^2
  \]

- Now, if \( f_2 \) is continuous on bounded \( I \), then for every closed interval \( I' \subseteq I \) (which is also bounded). The Cauchy problem admits a unique solution on \( I' \).
  That means that the problem admits a unique solution on \( I(\forall x \in I, \exists I' \subseteq I, \text{closed, such that } x \in I') \).
[4.] (II) A unique global solution.

\( f \in C^1 \) so \( f \) is easily Lipschitz. Hence we have the existence and unicity of local solutions (on every closed interval of given length). We can then put together these solutions to get a global solution, which is unique.

[5.] (4.) \( y(x) = Ae^{3x^3} \)

You just have to check.

[6.] (2) \( y_0(9) = \sqrt{10} \)

\( y(x) = Ae^{\ln(x^2 + 9)} = \sqrt{x^2 + 9} \)

But \( y(0) = 3 \) \( \implies A = 1 \) \( \implies y(x) = \sqrt{x^2 + 9} \) hence \( y(9) = \sqrt{90} = 3\sqrt{10} \).

[7.] (3) \( y_4 = \frac{11}{4} \)

\( (x + 3)y' = y - 1, \quad y(1) = 2 \)

\[ y' = \frac{1}{x + 3} y - \frac{1}{x + 3} \]  

1. General solution of the homogeneous equation : \( y_0(x) = Ae^{\ln(x + 3)}, A \in \mathbb{R} \)

2. We let \( A \) vary in terms of \( x : A \rightarrow A(x), y(x) = A(x)(x + 3) \).

\( y'(x) = A'(x)(x + 3) + A(x) \) and we want (2) so

\( A'(x)(x + 3) = -\frac{1}{x + 3} \) \( \implies A(x) = \frac{1}{x + 3} + c, c \in \mathbb{R} \).

3. Finally, we find \( c \) to obtain \( y_1(1) = 2 \) where \( y_1(x) = (\frac{1}{x + 3} + c)(x + 3) = 1 + c(x + 3) \)

\( 2 = y_1(1) = 1 + 4c \) \( \implies c = \frac{1}{4} \).

Hence : \( y_1(x) = 1 + \frac{4}{4}x + \frac{3}{4} \) and \( y_1(4) = \frac{11}{4} \)

[8.] (2) \( y = e^{\pm \sqrt{2x^3 + c}} \).

We have to change the equation a little bit : \( y' = \frac{5x^4 y}{\ln(y)} \) \( \iff \frac{y'}{y} \ln(y) = 5x^4 \) \( \iff (\ln(y))' \ln(y) = 5x^4 \)

Let \( Y := \ln(y) \) and the system becomes : \( Y'Y = 5x^4 \)

\( \iff (\frac{1}{2}Y)^2 = 5x^4 \) \( \iff \frac{1}{2}Y^2 = x^5 + c, c \in \mathbb{R} \).

\( \iff Y = \pm \sqrt{2x^3 + c}, c \in \mathbb{R} \) but we know that \( y = e^Y \)

\( \iff y = e^{\pm \sqrt{2x^3 + c}} \).

[9.] (1) \( u = -7 + Ce^{\frac{1}{2}t^2 + 6t} \)

\( u' = (6 + t)u + 7t + 42 \)

1. We solve the homogeneous equation to find the solution \( u_0 : u_0(t) = Ae^{\frac{1}{2}t^2 + 6t} \)
2. We vary the constant $A$ in terms of $t$: $u_1(t) = A(t)e^{\frac{1}{2}t^2+6t}$

$u_1'(t) = A'(t)e^{\frac{1}{2}t^2+6t} + (6 + t)u_1(t)$

By (3): $A'(t) = (7t + 42)e^{-\frac{1}{2}t^2-6t} \Rightarrow A(t) = -7e^{-\frac{1}{2}t^2-6t} + c, c \in \mathbb{R}$

Then we get our answer: $u_1(t) = -7 + ce^{\frac{1}{2}t^2+6t}, c \in \mathbb{R}$.