1. Answer (iv).

• By a direct calculation:

\[ w(x, y, z) = w(t^2, 1 - t, 1 + 3t) = t^2 e^{\frac{1}{1+3t}}. \]

Therefore,

\[ \frac{dw}{dt} = 2te^{\frac{1}{1+3t}} + t^2e^{\frac{1}{1+3t}} \left( \frac{1 - t}{1 + 3t} \right)' \]

\[ = 2te^{\frac{1}{1+3t}} + t^2e^{\frac{1}{1+3t}} \left( \frac{-(1 + 3t) - 3(1 - t)}{(1 + 3t)^2} \right) \]

\[ = e^{\frac{z}{y}} \left( 2t + x \left( \frac{-z - 3y}{z^2} \right) \right) \]

\[ = e^{\frac{z}{y}} \left( 2t - \frac{x - 3yx}{z^2} \right). \]

• We can also use the chain rule as following: let \( \kappa(t) = (t^2, 1 - t, 1 + 3t) \). Then \( w(t) = w(\kappa(t)) \) and thus

\[ w'(t) = \langle \nabla w(\kappa(t)), \kappa'(t) \rangle \]

\[ = \left( \left( e^{\frac{z}{y}}, \frac{x}{z} e^{\frac{z}{y}}, \frac{-xy}{z^2} e^{\frac{z}{y}} \right)^T, (2t, -1, 3)^T \right) \]

\[ = e^{\frac{z}{y}} 2t + \frac{x}{z} e^{\frac{z}{y}} (-1) - \frac{xy}{z^2} e^{\frac{z}{y}} \]

\[ = e^{\frac{z}{y}} \left( 2t - \frac{x - 3yx}{z^2} \right). \]

2. Answer (iv). The equation of the hyperplane is given at \( a \) by

\[ z = \nabla f(a) \cdot (x - a) + f(a). \]

Applying this formula for \( a = (-3, 2) \), we get

\[ z = 2 + (1, -2) \cdot (x - (-3), y - 2) = 2 \iff z = 2 + (1, -2) \cdot (x + 3, y - 2) \]

\[ \iff z = 2 + x + 3 - 2(y - 2) \]

\[ \iff z = 9 + x - 2y. \]

Remark: It may be a good thing to check that \((-3, 2, 2)\) verify the equation of the hyperplane that we have found to be sure we didn’t do any mistakes in our calculation.
3. Answer (ii). The linear approximation at \( \mathbf{a} \) is given by

\[
f(x) = \nabla f(\mathbf{a}) \cdot (x - \mathbf{a}) + f(\mathbf{a}).
\]

We have that

\[
\frac{\partial f}{\partial x}(x, y) = \frac{\partial}{\partial x} \sqrt{7 - x^2 - 2y^2} = \frac{-x}{\sqrt{7 - x^2 - y - 2}}
\]

\[
\frac{\partial f}{\partial y}(x, y) = \frac{\partial}{\partial y} \sqrt{7 - x^2 - 2y^2} = \frac{-2y}{\sqrt{7 - x^2 - 2y^2}}.
\]

Therefore,

\[
\nabla f(2, -1) = (-2, 2).
\]

Applying the formula above with \( \mathbf{a} = (2, -1) \) we get

\[
z = 7 + 2y - 2x.
\]

4. Answer (iii). Let first observe that

\[
\frac{\partial r}{\partial x} = \frac{x}{r} \quad \text{and} \quad \frac{\partial r}{\partial y} = \frac{y}{r}.
\]

Therefore

\[
\frac{\partial}{\partial x} \left( \frac{1}{r} \right) = \frac{-x}{r^3}
\]

and thus

\[
\frac{\partial^2}{\partial x^2} \left( \frac{1}{r} \right) = \frac{\partial}{\partial x} \left( -\frac{x}{r^3} \right) = \frac{-r^3 + x \cdot 3r^2 \cdot \frac{x}{r}}{r^6} = \frac{3x^2 - r^2}{r^5}
\]

and by an argument of symmetry, we also have

\[
\frac{\partial^2}{\partial y^2} \left( \frac{1}{r} \right) = \frac{3y^2 - r^2}{r^5}.
\]

Finally

\[
\frac{\partial^2}{\partial x \partial y} \left( \frac{1}{r} \right) = \frac{\partial}{\partial y} \left( \frac{y}{r^3} \right) = \frac{3xy}{r^5} = \frac{\partial^2}{\partial y \partial x} \left( \frac{1}{r} \right),
\]

and thus

\[
\text{Hess} \left( \frac{1}{r} \right)(x, y) = \frac{1}{r^5} \begin{pmatrix} \frac{3x^2 - r^2}{3xy} & \frac{3xy}{3y^2 - r^2} \\ \frac{3xy}{3y^2 - r^2} & \frac{3xy}{3y^2 - r^2} \end{pmatrix}.
\]

For \((x, y) = (3, 4)\) we have \(r = 5\) and thus

\[
\text{Hess} \left( \frac{1}{r} \right)(3, 4) = \frac{1}{5^5} \begin{pmatrix} 2 & 36 \\ 36 & 23 \end{pmatrix}.
\]
5. Answer (iv). As the question 1, we’ll use the formula

\[
\frac{df(\kappa(t))}{dt} = \langle \nabla f(\kappa(t)), \kappa'(t) \rangle,
\]

for \( f(x, y) = x \ln(x + 11y) \) and \( \kappa(t) = (\sin(t), \cos(t)) \). We have that

\[
\nabla f(x, y) = \left( \ln(x + 11y) + \frac{x}{x + 11y}, \frac{11x}{x + 11y} \right),
\]

\[
\kappa'(t) = (\cos(t), -\sin(t))
\]

and thus

\[
h'(t) = \langle \nabla f(\kappa(t)), \kappa'(t) \rangle
\]

\[
= \left( \ln(x + 11y) + \frac{x}{x + 11y} \right) \cos(t) + \frac{11x}{x + 11y}(-\sin(t))
\]

\[
= \ln(x + 11y) \cos(t) + \frac{x}{x + 11y}(\cos(t) - 11 \sin(t)).
\]

6. Answer (iii). For the first assertion,

\[
D_x f(0, 0) = \lim_{x \to 0} \frac{f(x, 0) - f(0, 0)}{x} = \lim_{x \to 0} \frac{f(x, 0)}{x} = \lim_{x \to 0} \frac{x^3 \cdot 0}{x^2 + 0^2} = 0,
\]

and

\[
D_y f(0, 0) = \lim_{y \to 0} \frac{f(0, y) - f(0, 0)}{y} = \lim_{y \to 0} \frac{0y}{0 + y^2} = 0.
\]

Therefore

\[
D_x f(0, 0) = D_y f(0, 0) = 0.
\]

For the second assertion, we have that

\[
\nabla f(x, y) = (D_x f, D_y f) = \left( \frac{3x^2 y}{x^2 + y^2} - \frac{2x^4 y}{(x^2 + y^2)^2}, \frac{x^3}{x^2 + y^2} - \frac{2x^3 y^2}{(x^2 + y^2)^2} \right).
\]

Therefore

\[
D_{xy} f(0, 0) = \lim_{x \to 0} \frac{D_y(x, 0) - D_y(0, 0)}{x} = \lim_{x \to 0} \frac{x}{x^2} = 1,
\]

and

\[
D_{yx} f(0, 0) = \lim_{y \to 0} \frac{D_x(0, y) - D_x(0, 0)}{y} = 0.
\]

Therefore

\[
D_{xy} f(0, 0) \neq D_{yx} f(0, 0).
\]

Remark: The fact that \( D_{xy} f(0, 0) \neq D_{yx} f(0, 0) \) tell us that \( f \) is not \( C^2 \) at \( 0 \).
7. Answer (iv). In the previous question, we compute $D_x f(0,0) = D_y f(0,0) = 0$. Since the partial derivative are continuous at 0, we have that

$$\nabla f(0) = 0.$$ 

By the way, the normal orientation is (0, 0, 1) and thus, the equation of the hyperplane is given by $z = 0$.

8. Answer (ii). We have that

$$\frac{\partial^2}{\partial x_i \partial x_j} (fg) = \frac{\partial}{\partial x_i} \left( g \frac{\partial f}{\partial x_j} + f \frac{\partial g}{\partial x_j} \right)$$

$$= \frac{\partial g}{\partial x_i} \frac{\partial f}{\partial x_j} + g \frac{\partial^2 f}{\partial x_i \partial x_j} + \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} + f \frac{\partial^2 g}{\partial x_i \partial x_j}.$$ 

Therefore,

$$\text{Hess}(fg)(\bar{x})_{ij} = g\text{Hess}(f)(\bar{x})_{ij} + f\text{Hess}(f)(\bar{x})_{ij} + \nabla f(\bar{x}) \nabla g(\bar{x})^T_{ij} + \nabla g(\bar{x}) \nabla f(\bar{x})^T_{ij},$$

and the claim follow.