1. Answer (iv). The Jacobian is given by

\[
J_v(x) = \begin{pmatrix}
\frac{\partial}{\partial x} xy^2 & \frac{\partial}{\partial y} xy^2 \\
\frac{\partial}{\partial x} xy^3 & \frac{\partial}{\partial y} xy^3 \\
\frac{\partial}{\partial x} x & \frac{\partial}{\partial y} x
\end{pmatrix} = \begin{pmatrix} y^2 & 2xy \\
y^3 & 3xy \\
1 & 0
\end{pmatrix}.
\]

2. Answer (i). A simple computation gives

\[
\nabla \times F(x, y, z) = \begin{pmatrix}
\frac{\partial}{\partial y} z^3 - \frac{\partial}{\partial z} y^2 \\
\frac{\partial}{\partial z} x - \frac{\partial}{\partial x} z^3 \\
\frac{\partial}{\partial x} y^2 - \frac{\partial}{\partial y} x
\end{pmatrix} = 0.
\]

3. Answer (iv). The gradient is defined for functions with real values only.

4. Answers (i), (ii) and (iii). Let’s prove it!

(i) Since \( \text{Hess}(f)_{ii} = \frac{\partial^2 f}{\partial x^2} \), the claim follows.

(ii)

\[
\nabla \cdot \nabla f(x) = \begin{pmatrix}
\frac{\partial}{\partial x_1} f(x) \\
\vdots \\
\frac{\partial}{\partial x_n} f(x)
\end{pmatrix} \cdot \begin{pmatrix}
\frac{\partial}{\partial x_1} f(x) \\
\vdots \\
\frac{\partial}{\partial x_n} f(x)
\end{pmatrix} = \sum_{i=1}^n \frac{\partial^2 f}{\partial x_i^2}(x).
\]

(iii) It’s the definition of the Laplacian.

5. Answer (iv). Let \( \mathbf{v} = (v_1, v_2, v_3) \) and \( \mathbf{w} = (w_1, w_2, w_3) \). We have that

\[
\mathbf{v} \times \mathbf{w} = \begin{pmatrix} v_2 w_3 - v_3 w_2 \\
v_3 w_1 - v_1 w_3 \\
v_1 w_2 - v_2 w_1
\end{pmatrix}.
\]

We can see that (ii) can’t be correct since there is no term of the form \( D_2v_3 \cdot D_3w_3 \). Also, (iii) is not defined since \( \nabla \cdot \mathbf{v} \) is a scalar and not a vector. Then, the only possibility is (i). Let show that it’s not correct.

\[
\text{div}(\mathbf{v} \times \mathbf{w}) = w_1(D_2v_3 - D_3v_2) - v_1(D_2w_3 - D_3w_2) + w_2(D_3v_1 - D_1v_3) - v_2(D_3w_1 - D_1w_3) + w_3(D_1v_2 - D_2v_1) - v_3(D_1w_2 - D_2w_1) = \langle \nabla \times \mathbf{v}, \mathbf{w} \rangle - \langle \mathbf{v}, \nabla \times \mathbf{w} \rangle.
\]
6. Answer (iv). We have that

\[ \nabla(v \cdot w) = \nabla \left( \sum_{i=1}^{n} v_i w_i \right) \]

\[ = \left( \sum_{i=1}^{n} w_i D_1 v_i \right) + \left( \sum_{i=1}^{n} v_i D_1 w_i \right) \]

\[ = F_v^T w + F_w^T v \]

\[ = (w^T F_v)^T + (v^T F_w)^T \]

\[ = (w^T F_v + v^T F_w)^T \]

7. Answer (iv). Recall that

\[ F_w(0, -2) = \left( F_v(1, -1) \right)^{-1} \]

However

\[ F_v(x, y) \big|_{(x,y)=(1,-1)} = \begin{pmatrix} 3x^2 & -2y \\ -2x & 3y^2 \end{pmatrix} \bigg|_{(x,y)=(1,-1)} = \begin{pmatrix} 3 & 2 \\ -2 & 3 \end{pmatrix}, \]

and thus

\[ F_w(0, -2) = \frac{1}{13} \begin{pmatrix} 3 & -2 \\ 2 & 3 \end{pmatrix}. \]

8. Answer (ii). For the first assertion,

\[ F_v(x, y) = \begin{pmatrix} 2x - 2y & -2x \\ 2y & 2x - 2y \end{pmatrix} \bigg|_{(x,y)=(1,1)} = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}, \]

and thus \( \det F_v(1, 1) = 4 \). For the second assertion, let \( v \in C^1(\mathbb{R}^2) \) locally invertible at \( x_0 \). Let denote \( w \) it’s inverse in a neighborhood of \( x_0 \). If \( w \in C^1 \), then \( \det F_w(x_0) \neq 0 \) (see lecture). On the other hand, if \( w \notin C^1 \), we can’t say anything. Indeed, \( v(x,y) = (x^3, y^3) \) is in \( C^1(\mathbb{R}^2) \), it’s inverse is \( w(s,t) = (s^{1/3}, t^{1/3}) \) but \( \det F_v(0,0) = 0 \).

Remark: If \( v \) is an invertible vector field that is \( C^1 \) and if it’s inverse is also \( C^1 \), then \( v \) is called a diifeomorphism.

9. Answer (iv). We use the formula

\[ Q(x, y) = f(0,0) + x D_x f(0,0) + y D_y f(0,0) + \frac{x^2}{2} D_{xx} f(0,0) + x y D_{xy} f(0,0) + \frac{y^2}{2} D_{yy} f(0,0). \]

We find easily

\[ \nabla f(x, y) = ( - \sin(x-y) + 2 \cos(x-y), \sin(x-y) - 2 \cos(x-y) ) \]
and thus
\[ \nabla f(0, 0) = (2, -2). \]

Also
\[
\text{Hess}(f)(x, y) = \begin{pmatrix}
-\cos(x - y) - 2\sin(x - y) & \cos(x - y) + 2\sin(x - y) \\
\cos(x - y) + 2\sin(x - y) & -\cos(x - y) - 2\sin(x - y)
\end{pmatrix}
\]

and thus
\[
\text{Hess}(f)(0, 0) = \begin{pmatrix}
-1 & 1 \\
1 & -1
\end{pmatrix}.
\]

Replacing all those values in the formula above allow us to conclude.

10. Answer (i). We use the same formula as the previous exercise and by computation, we find
\[ \nabla f(0, 0) = (-2, -1) \]

and
\[
\text{Hess}(f)(0, 0) = \begin{pmatrix}
-4 & -2 \\
-2 & -1
\end{pmatrix}.
\]

Details of calculation are left to the reader.