1. Answer (i). Since 
\[ \nabla f(x, y) = (2xe^x, \cos(y)), \]
we have that \( \nabla f(x, y) \neq 0 \) for all \( x, y \in U \). Indeed, \( \nabla f(x, y) = 0 \) only if \( x = 0 \) and \( y = \frac{\pi}{2} + k\pi \). But since \( (0, \frac{\pi}{2} + k\pi) \notin B(0, \frac{\pi}{4}) \) the gradient is non zero in \( U \). It’s not injective because \( f(x, y) = f(-x, y) \) for all \( (x, y) \in U \).

2. Answer (i). Let denote 
\[ v_1(x, y) = \cos \left( \frac{x}{\sqrt{x^2 + y^2}} + \cos \left( \sqrt{x^2 + y^2} \right) \right) \quad \text{and} \quad v_2(x, y) = e^{\frac{1}{\sqrt{x^2 + y^2}}}. \]

We have that
\[
\begin{align*}
\frac{\partial v_1}{\partial x} &= -\sin\left( \frac{x}{\sqrt{x^2 + y^2}} + \cos\left( \sqrt{x^2 + y^2} \right) \right) \left( \frac{y^2}{(x^2 + y^2)^{3/2}} - \sin\left( \sqrt{x^2 + y^2} \right) \cdot \frac{x}{\sqrt{x^2 + y^2}} \right), \\
\frac{\partial v_1}{\partial y} &= \sin\left( \frac{x}{\sqrt{x^2 + y^2}} + \cos\left( \sqrt{x^2 + y^2} \right) \right) \left( \frac{xy}{(x^2 + y^2)^{3/2}} + \sin\left( \sqrt{x^2 + y^2} \right) \cdot \frac{y}{\sqrt{x^2 + y^2}} \right), \\
\frac{\partial v_2}{\partial x} &= -\frac{1}{(x^2 + y^2)^{3/2}} e^{1/\sqrt{x^2 + y^2}}, \\
\frac{\partial v_2}{\partial y} &= -\frac{1}{(x^2 + y^2)^{3/2}} e^{1/\sqrt{x^2 + y^2}},
\end{align*}
\]
and thus
\[
\det J_{\bar{v}}(1, 1) =
\begin{vmatrix}
-\sin\left( \frac{1}{\sqrt{2}} + \cos\left( \sqrt{2} \right) \right) & \sin\left( \frac{1}{\sqrt{2}} + \cos\left( \sqrt{2} \right) \right) \\
-\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}
\end{vmatrix}
= \frac{e^{1/\sqrt{2}}}{4} \sin\left( \frac{1}{\sqrt{2}} + \cos\left( \sqrt{2} \right) \right) \neq 0,
\]
because \( \sqrt{2} \in \left( 1, \frac{\pi}{2} \right) \) (since \( \sqrt{2} \approx 1.4 \) and \( \frac{\pi}{2} \approx 1.57 \)) and
\[
\frac{1}{\sqrt{2}} \leq \frac{1}{\sqrt{2}} + \cos\left( \sqrt{2} \right) \leq 2 < \frac{3\pi}{4}.
\]
It’s not invertible avec all \( \mathbb{R}^2 \setminus \{(0, 0)\} \) since for example \( f(0, 1) = f(0, -1) \), and thus not one-to-one over all \( \mathbb{R}^2 \setminus \{(0, 0)\} \).
3. Answer (i). Let denote $\bar{v}(x, y) = (v_1(x, y), v_2(x, y))$. Then $\bar{v}^{-1} = (v_1^{-1}, v_2^{-1})$. Finally,
\[ g(s, t) = f(\bar{v}^{-1}(s, t)) = v_1^{-1}(s, t)v_2^{-1}(s, t). \]
Therefore
\[
\frac{\partial g}{\partial s} = v_2^{-1}(s, t) \frac{\partial v_1^{-1}}{\partial s} + v_1^{-1} \frac{\partial v_2^{-1}}{\partial s}, \\
\frac{\partial g}{\partial t} = v_2^{-1}(s, t) \frac{\partial v_1^{-1}}{\partial t} + v_1^{-1} \frac{\partial v_2^{-1}}{\partial t}.
\]
We know have to compute de Jacobian matrix of $\bar{v}^{-1}$ at $(e, 3)$. First remark that $\bar{v}^{-1}(e, 3) = (1, 1)$. We compute the Jacobian of $\bar{v}$ at $(1, 1)$. It’s given by
\[
\mathcal{J}_\bar{v}(1, 1) = \left( \begin{array}{cc} y e^{xy} & xe^{xy} \\ 1 + 2x & 2y \end{array} \right) \bigg|_{(1, 1)} = \left( \begin{array}{cc} e & e \\ 3 & 2 \end{array} \right),
\]
and thus
\[
\mathcal{J}_{\bar{v}^{-1}}(e, 3) = \left( \mathcal{J}_\bar{v}(1, 1) \right)^{-1} = -\frac{1}{e} \left( \begin{array}{cc} 2 & -e \\ -3 & e \end{array} \right).
\]
We finally conclude that
\[
\frac{\partial g}{\partial s}(e, 3) = 1 \cdot \frac{2}{-e} + 1 \cdot \left( \begin{array}{c} -3 \\ -e \end{array} \right) = e^{-1}, \\
\frac{\partial g}{\partial t}(e, 3) = 1 \cdot \left( \begin{array}{c} -e \\ -e \end{array} \right) + 1 \cdot \left( \begin{array}{c} e \\ -e \end{array} \right) = 0,
\]
and thus
\[
\nabla_{st}g(e, 3) = \left( e^{-1}, 0 \right).
\]
4. Answer (ii). The function $f$ is $C^1$. We have that
\[
D_x f(0, 0) = (2x + y\cos(xy))|_{(0, 0)} = 0,
\]
and
\[
D_y f(0, 0) = (2e^y + x\cos(xy))|_{(0, 0)} = 2 \neq 0.
\]
The implicit function theorem allow us to conclude on the existence. Moreover
\[
\varphi'(0) = -\frac{D_x f(0, \varphi(0))}{D_y f(0, f(0))} = -\frac{0}{2} = 0.
\]
5. Answer (i). We have that $f(0, 0, 1) = 0$ and that $f$ is $C^1$. We have that
\[
\frac{\partial f}{\partial z} f(0, 0, 1) = (xe^y + 4z^3e^{xy} + z^4xe^{xz})|_{(0,0,1)} = 4 \neq 0.
\]
Therefore, by the implicit function theorem, there is an implicit function $z = \varphi(x, y)$ in a neighborhood of $(0, 0, 1)$ with $\varphi(0, 0) = 1$. Moreover
We conclude that the equation of the tangent plane at \((0, 0, 1)\) is given by \(z = 1\).

6. Answer (iii).

\[
2x^3 - y^3 + 9xy + 1 = 0 \implies \frac{d}{dx} (2x^3 - y^3 + 9xy + 1) = 0
\]
\[
\implies 6x^2 - 3y^2 \frac{dy}{dx} + 9y + 9x \frac{dy}{dx} = 0
\]
\[
\implies \frac{dy}{dx} (9x - 3y^2) = -9y - 6x^2
\]
\[
\implies \frac{dy}{dx} = \frac{2x^2 + 3y}{y^2 - 3x}.
\]

7. Answer (ii). As previously

\[
xe^{2y} - yz + ze^{3x} = 0 \implies \frac{\partial}{\partial x} (xe^{2y} - yz + ze^{3x}) = 0
\]
\[
\implies e^{2y} - y \frac{\partial z}{\partial x} + \frac{\partial z}{\partial x} e^{3x} + 3ye^{3x} = 0
\]
\[
\implies \frac{\partial z}{\partial x} (e^{3x} - y) = -(e^{2y} - 3ze^{3x})
\]
\[
\implies \frac{\partial z}{\partial x} = \frac{-e^{2y} + 3ye^{3x}}{e^{3x} - y}.
\]

8. Answer (i). We use the rule

\[
\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial \theta},
\]

where

\[
\frac{\partial x}{\partial \theta} = -2r \sin(2\theta) \quad \text{and} \quad \frac{\partial y}{\partial \theta} = 2r \cos(\theta).
\]

The claim follow.

9. Answer (iv). Even if we can’t use the implicit function theorem (since \(D_x g(0, 0) = D_y g(0, 0) = 0\)) the function defined by \(\varphi_2(y) = y^3\) verify the wanted properties.
10. Answer (iv). Since

\[ D_x f(0, 0) = (e^{x-\alpha y^2})_{(0,0)} = 1 \neq 0, \]

for all \( \alpha \in \mathbb{R} \) and

\[ D_y f(0, 0) = 0 \]

for all \( \alpha \in \mathbb{R} \) (because if \( \alpha \neq 0 \) we also have that \( (-2\alpha y^2 e^{x-\alpha y^2})_{(0,0)} = 0 \)). Since \( f \)

is \( \mathcal{C}^1 \), we can use the implicit function theorem and conclude.