[1.](ii) TF. $\nabla f = (-2, 3, 6)^T \neq 0$. So the extrema lie on the boundary. We have to maximize:

$$
\begin{aligned}
&\begin{cases}
-2x + 3y + 6z + 42 \\
\text{Subject to } x^2 + \frac{3y^2}{2} + 3z^2 + 2x - 6z = \frac{3}{2}
\end{cases}
\end{aligned}
$$

$$\nabla g = \begin{pmatrix} 2x + 2 \\ 3y \\ 6z - 6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}
$$

But $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ is not in $M = \{ \bar{x} : a(\bar{x}) = \frac{3}{2} \}$. So the extrema satisfy:

$$\nabla f + \lambda \nabla g = \bar{0} = \begin{pmatrix} -2 \\ 3 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} 2x + 2 \\ 3y \\ 6z - 6 \end{pmatrix}
$$

$$\Rightarrow \lambda = \frac{1}{x+1} = -\frac{1}{y} = -\frac{1}{z-1} \Rightarrow \begin{cases} x = \frac{1}{\lambda} - 1 \\ y = -\frac{1}{\lambda} \\ z = -\frac{1}{\lambda} + 1 \end{cases}
$$

$$g(x, y, z) = \frac{1}{\lambda^2} - \frac{1}{\lambda} + 1 + \frac{3}{2\lambda^2} + 3\left(\frac{1}{\lambda^2} - \frac{2}{\lambda} + 1\right) + \frac{2}{\lambda} - 2 + \frac{6}{\lambda} - 6 = \frac{\sqrt{5}}{\lambda^2} - 4 = \frac{3}{2} \Rightarrow \lambda = \pm 1$$

The extrema are:

$$\begin{cases} 
\lambda = 1 \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \Rightarrow f = 39 \text{ Minimum}
\end{cases}
$$

$$\begin{cases} 
\lambda = -1 \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} \Rightarrow f = 61 \text{ Maximum}
\end{cases}
$$

[2.](iv) $\nabla f = (ye^{(x-1)y}, (x - 1)e^{(x-1)y})^T = \bar{0}$ only if $y = 0, x = 1$ so the extrema are on $\partial E$. Using the parametrization $x = \cos(\theta), y = \sin(\theta)$ we reduce the problem to maximizing/minimizing $(\cos(\theta) - 1)\sin(\theta) = g(\theta)$.

$$g'(\theta) = -\sin^2(\theta) + \cos^2(\theta) - \cos(\theta) = 2\cos^2(\theta) - \cos(\theta) - 1 = (2\cos(\theta) + 1)(\cos(\theta) - 1) \quad (\cos(\theta) = 1 \Rightarrow e^{(x-1)y} = e^0 = 1)
$$

$$\cos(\theta) = -\frac{1}{2} \Rightarrow x = -\frac{1}{2}, y = \pm \frac{\sqrt{3}}{2}$$

$$\begin{cases} 
 f\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right) = e^{\frac{3\sqrt{3}}{4}} > 1 \\
 f\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right) = e^{-\frac{3\sqrt{3}}{4}} < 1
\end{cases} \Rightarrow (iv).$$
Let's apply the change \((x, y, z) = (\cos(\theta), \sin(\theta))\):

\[
f(x, y, z) = \frac{1}{2} - \cos(\theta)\sin(\theta) = g(\theta)
g'(\theta) = \sin^2(\theta) - \cos^2(\theta) = 0\] for \(\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\)

For \(\theta = \frac{\pi}{4}, \frac{5\pi}{4}\), \(\cos(\theta)\sin(\theta) = \frac{1}{2} \Rightarrow f = -\frac{1}{4} \Rightarrow \text{Minimum.}

For \(\theta = \frac{3\pi}{4}, \frac{7\pi}{4}\), \(\cos(\theta)\sin(\theta) = -\frac{1}{2} \Rightarrow f = \frac{1}{4} \Rightarrow \text{Maximum.}

[4.](i) \(g(x, y) = e^{x^2} + e^{y^2} - 4 \Rightarrow \nabla g(x, y) = (2xe^{x^2}, 2ye^{y^2}) = 0\) only if \((x, y) = (0,0)\) where the constraint is not satisfied. The Lagrange approach is then justified:

\[
\nabla f + \lambda \nabla g = \left(\frac{-xe^{-x^2}}{ye^{-y^2}}\right) + \lambda \left(\frac{2xe^{x^2}}{2ye^{y^2}}\right) = 0
\]

\[
\begin{align*}
\lambda &= -\frac{1}{2}e^{-2x^2} \text{ or } x = 0 \\
\lambda &= \frac{1}{2}e^{-2y^2} \text{ or } y = 0
\end{align*}
\]

\[
\begin{align*}
x = 0 : e^{y^2} = 4 - 1 &= 3 \Rightarrow y = \pm \sqrt{(\log(3))} \\
y = 0 : e^{x^2} = 4 - 1 &= 3 \Rightarrow x = \pm \sqrt{(\log(3))}
\end{align*}
\]

If \(x = 0, f(0, \pm \sqrt{(\log(3))}) = \pm \frac{1}{2}(1 - e^{\log(3)}) = \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}
\]

If \(x = 0, f(\pm \sqrt{(\log(3)), 0}) = -\frac{1}{2}(1 - e^{\log(3)}) = -\frac{1}{3}
\]

[5.](iii) Let \(g_1(x, y, z) = x^2 + y^2 + 2z^2 - 4\) and \(g_2(x, y, z) = xyz - 1\). We want to find the extrema of \(f(x, y, z) = x^2 + y^2 + z^2\) subject to \(g_1 = g_2 = 0\)

Remark: If \((x, y, z)\) is an extremum, then so are \((-x, -y, z), (x, -y, -z), (-x, y, -z)\) because \(f, g_1, g_2\) remain unchanged. So then for an extremum, we can suppose that \(x\) and \(y\) are positive which implies that \(z\) is positive as well (because \(xyz = 1 > 0\)). We are now looking for positive \(x, y\) and \(z\):

First list: \(\nabla g_1 = \left(\begin{array}{c} 2x \\ 2y \\ 4z \\
\end{array}\right), \nabla g_2 = \left(\begin{array}{c} yz \\ xz \\ xy \\
\end{array}\right)\)

If \(\nabla g_1\) and \(\nabla g_2\) are linearly independent (as it is impossible that either one of them is equal to 0), we have:

\[
\begin{align*}
\begin{bmatrix} 2x \\ 2y \\ 4z \end{bmatrix} &= c \begin{bmatrix} yz \\ xz \\ xy \end{bmatrix} \Rightarrow x = \frac{2x}{yz} = \frac{2y}{xz} = \frac{4z}{xy} \Rightarrow x = y = z \\
x^2 &= y^2 \Rightarrow x = y \because x, y > 0
\end{align*}
\]

\[
ge_2 = 0 \Rightarrow x^2 + y^2 + 2z^2 = 3x^2 = 4 \Rightarrow x = y = \frac{2}{\sqrt{3}} \text{ and } z = \frac{\sqrt{7}}{\sqrt{3}} \Rightarrow xyz = \frac{4\sqrt{7}}{3\sqrt{3}} \neq 1.
\]

So \(\nabla g_1\) and \(\nabla g_2\) are linearly dependent, which means that the first list is empty.
Second list: We’re looking for \( x, y, z > 0 \) with 
\[
\nabla f = \begin{pmatrix} 2x \\ 2y \\ 2z \\ 4z \\ xy \\ yz \\ xz \end{pmatrix} = \lambda \begin{pmatrix} 2x \\ 2y \\ 2z \\ 4z \\ xy \\ yz \\ xz \end{pmatrix} + \sigma \begin{pmatrix} yz \\ xz \\ xy \end{pmatrix}
\]
Then 
\[
\begin{cases}
2x - 2\lambda x = \sigma yz \\
2y - 2\lambda y = \sigma xz \\
x + y + z = \sigma xyz = (x y - 2\lambda x)x = (2 - 2\lambda)x^2 \\
y + z = (2y - 2\lambda y)y = (2 - 2\lambda)y^2 \\
z = (x - 2\lambda x)z = (2 - 2\lambda)z^2
\end{cases}
\]
We notice that \( \lambda = 1 \) is impossible because it would imply that \( \sigma = 0 \) and hence that \( z = 0 \) which violates the condition \( g_2 = 0 \). Which means that \( x^2 = y^2 \Rightarrow x = y \) as they are both positive.

So \( x^2 + y^2 + 2z^2 - 4 = 0 \) \( \Rightarrow \) \( 2x^2 + 2z^2 - 4 = 0 \) \( \iff \) \( z^3 - 2z + 1 = 0 \) \( \iff \) \( (z - 1)(z^2 + z - 1) = 0 \) \( \Rightarrow \) \( z = 1 \) or \( z = -1 \pm \sqrt{2} \)

We know that \( z \) is positive so 2 choices:
\[
z = 1 \Rightarrow x^2 = y^2 = \frac{5 - 1 + \sqrt{5}}{2} = \frac{\sqrt{5} + 1}{2}
\]
\[
x^2 + y^2 + z^2 = 3 \text{ in the first case and } x^2 + y^2 + z^2 = \sqrt{5} + 1 + \left( \frac{-1 + \sqrt{5}}{2} \right)^2 = \frac{5 + \sqrt{5}}{2} > 3 \text{ in the second case.}
\]
So the first case is a minimum and the second case is a maximum.

[6.] (iv) We have to maximize \( f = \prod_{k=1}^n x_k^2 \) subject to \( g = \sum_{k=1}^n x_k^2 = 1 \).
\( \nabla g = 2\bar{x} = 0 \iff \bar{x} = 0 \) which is not the maximum and doesn’t satisfy the condition because \( g(\bar{0}) = 0 \neq 1 \). Then we can use the Lagrange approach: We are looking for \( \bar{x}, \lambda \) such that:
\[
\nabla f + \lambda \nabla g = 0 \text{ which means } \forall k, \ 2x_n \prod_{j \neq k} x_j^2 + 2\lambda x_n = 0 . \text{ As we can’t have } x_n = 0 \text{ for the maximum, these equations imply:} \\
\lambda = -\frac{1}{2} \prod_{j \neq k} x_j^2 \Rightarrow \frac{x_k^2}{f(x)} , \text{ which means that } x_k^2 = x_j^2 \ \forall j, k \text{ hence } \forall n, x_n^2 = \frac{1}{n}
\]

[7.] (iii) We can see that \( \forall x, y, z \) on the ellipsoid \( \| x \|, \| y \|, \| z \| \leq 10^5 \) so \( x + y + z \leq 3 \times 10^5 < c10^8 \) so the hyperplane doesn’t intersect the ellipsoid and generally there is only one minimum. The distance of a point \( (x,y,z) \) on the hyperplane to the ellipsoid is given by: \( \| \frac{x+y+z}{\sqrt{3}} - 10^8 \| \) . As we have that \( x + y + z \leq 3.10^5 < c10^8 \), we have to minimize \( 10^8 - \frac{x+y+z}{\sqrt{3}} \) which is just about maximizing \( f(x, y, z) := x + y + z \) subject to 
\[
g(x, y, z) := x^2 + y^2 + \frac{10^9}{9} z^2 = 10^{10} \nabla g = \begin{pmatrix} 2x \\ 2y \\ 20g \end{pmatrix} = 0 \iff \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \bar{0} \text{ Which is impossible on the ellipsoid. Hence, the Lagrange approach works.} \\
\n\nabla f + \lambda \nabla g = \bar{0} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2x \\ 2y \\ 20g \end{pmatrix} \iff \lambda = -\frac{1}{2x} = -\frac{1}{2y} = -\frac{9}{20z} \iff x = y = \frac{10^9}{9} z
\]
And \( x^2 + y^2 + \frac{10^9}{9} z^2 = 2x^2 + \frac{10^9}{9} x^2 = 10^{10} \Rightarrow x = 10^5 \sqrt{\frac{10}{29}} = y, z = 9.10^4 \sqrt{\frac{10}{29}} \)
Which means \( x + y + z = 10^5 \sqrt{\frac{10}{29}} (1 + 1 + \frac{9}{10}) = 10^4 \sqrt{290} \).
The distance is: \( \frac{10^8 - 10^4 \sqrt{290}}{\sqrt{3}} = \frac{10^4(10^4 - \sqrt{290})}{\sqrt{3}} \)

[8.](i) We will rather look for the minimum of \( f^2(x, y) = x^2 + 3y^2 \) subject to \( x + y = 3 \).

Remark: For \( g = x + y - 3, \nabla g = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \). Which means the first list is empty, we will then consider the second list.

\[ \nabla (f^2) + \lambda \nabla g = 0 \iff \left( \begin{array}{c} 2x \\ 6y \end{array} \right) + \lambda \left( \begin{array}{c} 1 \\ 1 \end{array} \right) = 0 \iff \begin{cases} 2x + \lambda = 0 \\ 6y + \lambda = 0 \end{cases} \Rightarrow x = 3y \]

\[ x + y = 4y = 3 \Rightarrow y = \frac{3}{4}, x = \frac{9}{4} \]

\[ f(x, y) = \sqrt{x^2 + 3y^2} = \frac{1}{2} \sqrt{81 + 27} = \frac{3}{2} \sqrt{12} = \frac{3\sqrt{3}}{2} \]

Remark: When \( x, y \to \infty, f \to \infty \) so as \( f \) is continuous, the minimum exists and without calculation we can already rule out (iv). We can also consider \( x = 3 \) (and \( y = 0 \)) to see that the minimum is \( \leq 3 \) so (ii) isn’t possible.

[9.](ii) Just like [8.], the list 1 is empty and we look for \( x, y, \lambda \) with \( \begin{pmatrix} 3y \\ 3x \end{pmatrix} + \lambda \left( \begin{array}{c} 1 \\ 1 \end{array} \right) = 0 \)

\( \Rightarrow x = y = 4 \). And the maximum (it’s obviously a maximum) is equal to 48.

[10.](iv) We have to maximize \( xy \) subject to \( g(x, y) := \frac{x^2}{36} + \frac{y^2}{9} - 1 = 0 \)

\[ \nabla g = \left( \begin{array}{c} \frac{x}{18} \\ \frac{2y}{9} \end{array} \right) = 0 \iff (x, y) = (0, 0) \]

This violates the condition \( g(x, y) = 0 \) so the list 1 is empty. We then look for \( x, y, \lambda \) such that \( \begin{pmatrix} y \\ x \end{pmatrix} + \lambda \left( \begin{array}{c} \frac{1}{18} \\ \frac{2}{9} \end{array} \right) = 0 \iff \lambda = \frac{18y}{x} = \frac{9x}{2y} \Rightarrow 2y = x \)

And \( \frac{x^2}{36} + \frac{y^2}{9} = 1 \Rightarrow y = \frac{3}{\sqrt{2}}, x = 3\sqrt{2} \) and \( xy = 9 \)