Solutions to Exercise Session, May 2, 2016

1. Let $D = [0,1] \times [0,\pi/2]$. Calculate
\[
\iint_D \frac{x \sin y}{1 + x^2} \, dx dy.
\]
Solution.
\[
\iint_D \frac{x \sin y}{1 + x^2} \, dx dy = \int_0^1 \frac{x}{1 + x^2} \, dx \cdot \int_0^{\pi/2} \sin y \, dy
\]
\[
= \ln(1 + x^2) \bigg|_0^1 \cdot (-\cos y) \bigg|_0^{\pi/2}
\]
\[
= \frac{\ln 2}{2}.
\]

2. Let $D = [0,1] \times [1,2]$. Calculate
\[
\iint_D \frac{x}{x^2 + y^2} \, dx dy.
\]
Solution.
\[
\int_0^1 \frac{x}{x^2 + y^2} \, dx = \int_0^1 \frac{1}{2} \frac{d}{dx} \ln(x^2 + y^2) \, dx = \frac{1}{2} \left( \ln(1 + y^2) - \ln y^2 \right).
\]
and
\[
\frac{1}{2} \int_1^2 \ln(1 + y^2) \, dy = \frac{y \ln(1 + y^2)}{2} \bigg|_{y=1}^{y=2} - \int_1^2 \frac{y^2}{1 + y^2} \, dy = \frac{y \ln(1 + y^2)}{2} \bigg|_{y=1}^{y=2} - y + \arctan y \bigg|_{y=1}^{y=2},
\]
\[
-\frac{1}{2} \int_1^2 \ln(y^2) \, dy = y - y \ln y \bigg|_{y=1}^{y=2},
\]
where using $\arctan 1 = \pi/4$:
\[
\iint_D \frac{x}{x^2 + y^2} \, dx dy = \arctan 2 + \ln 5 - \frac{5 \ln 2}{2} - \frac{\pi}{4}.
\]

3. Let $D = [0,\pi] \times [0,1]$. Calculate
\[
\iint_D x \sin xy \, dx dy.
\]
Solution.

\[
\iint_D x \sin xy \, dx \, dy = \int_{0}^{\pi} \left( \int_{0}^{x} x \sin xy \, dy \right) \, dx \\
= \int_{0}^{\pi} (-\cos xy) \bigg|_{y=0}^{y=1} \, dx \\
= \int_{0}^{\pi} (1 - \cos x) \, dx \\
= (x - \sin x) \bigg|_{0}^{\pi} = \pi
\]

4. Let \( D \) be the interior of the triangle of summits \( A = (0,0) \), \( B = (\pi,0) \) and \( C = (\pi,\pi) \). Calculate

\[
\iint_D \cos(x + y) \, dx \, dy.
\]

Solution. Trivially \( D = \{(x, y) : 0 \leq y \leq x \leq \pi\} \). So

\[
\iint_D \cos(x + y) \, dx \, dy = \int_{0}^{\pi} \left( \int_{0}^{x} \cos(x + y) \, dy \right) \, dx \\
= \int_{0}^{\pi} (x \sin(x + y)) \bigg|_{y=0}^{y=x} \, dx \\
= \int_{0}^{\pi} x \sin 2x - x \sin x \, dx \\
= \int_{0}^{\pi} \frac{x}{2} (-\cos 2x)' + x(\cos x)' \, dx \\
= -\frac{3\pi}{2} + \int_{0}^{\pi} \frac{\cos 2x}{2} - \cos x \, dx \\
= -\frac{3\pi}{2}.
\]

5. Let \( D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\} \). Calculate the volume of

\[
E = \{(x, y, z) \in \mathbb{R}^3 : (x, y) \in D \text{ and } 0 \leq z \leq \sqrt{1 - x^2 - y^2}\}.
\]

Deduce the volume of the unit ball \( B_1(0) \) in \( \mathbb{R}^3 \).

Solution. Using the formula

\[
\int \sqrt{a^2 - x^2} \, dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \arcsin \frac{x}{a}, \ a > 0, \ |x| < |a|
\]
with \( a = \sqrt{1 - y^2} \) we get

\[
\text{Vol}(E) = \int_{-1}^{1} \left( \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \sqrt{1 - y^2 - x^2} \, dx \right) \, dy
\]

\[
= \int_{-1}^{1} \frac{x}{2} \sqrt{1 - y^2 - x^2} + \frac{1 - y^2}{2} \arcsin \frac{x}{\sqrt{1 - y^2}} \bigg|_{x=-\sqrt{1-y^2}}^{x=\sqrt{1-y^2}} \, dy
\]

\[
= \int_{-1}^{1} \frac{1 - y^2}{2} (\arcsin 1 - \arcsin(-1)) \, dy
\]

\[
= \frac{\pi}{2} \int_{-1}^{1} 1 - y^2 \, dy
\]

\[
= \frac{2\pi}{3}.
\]

The set \( E \) represents the half unit ball. So

\[
\text{Vol}(B_1(0)) = \frac{4\pi}{3}.
\]

6. Calculate

\[
\iint_{\mathbb{R}^2} e^{-|x-1|-|y|} \, dxdy.
\]

Solution.

\[
\iint_{\mathbb{R}^2} e^{-|x-1|-|y|} \, dxdy = \int_{-\infty}^{\infty} e^{-|x-1|} \, dx \cdot \int_{-\infty}^{\infty} e^{-|y|} \, dy
\]

\[
= \int_{-\infty}^{\infty} e^{-|t|} \, dt \cdot \int_{-\infty}^{\infty} e^{-|y|} \, dy
\]

\[
= 4 \int_{0}^{\infty} e^{-t} \, dt \cdot \int_{0}^{\infty} e^{-y} \, dy
\]

\[
= 4.
\]

7. True/False

(a) False. Without loss of generality assume that \((x_0, y_0) = (0, 0)\). Directional derivative of a function is given by \( \nabla f|_{(0,0)} \cdot \vec{v} \). Now consider,

\[
\vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos(0) \\ \sin(0) \end{pmatrix}
\]

By assumption we have,

\[
\nabla f|_{(0,0)} \cdot \vec{v} = 1
\]

and

\[
\nabla f|_{(0,0)} \cdot (-\vec{v}) = \nabla f|_{(0,0)} \cdot (\cos \pi, \sin \pi) = 1 \neq -\nabla f|_{(0,0)} \cdot \vec{v}
\]

Which is a contradiction. Note that if \( \vec{v} = (\cos \theta, \sin \theta) \) then \( \nabla f|_{(0,0)} \cdot \vec{v} \) has to be a linear function of \( \cos \theta \) and \( \sin \theta \) and in this case it is not.

(b) True. The Hessian is positive-definite.

(c) True. Because \( J_{vow} = J_v|_w \times J_W \).

(d) False. It is possible for a function to have the global minimum on the boundary of \( D \), but also a local minimum in the interior of \( D \), where then the hessian matrix is definite.
(e) True. Since this vector is in direction of $-\nabla f$.

(f) Solution.

i. True. Let us recall that a function \( f \) is differentiable at \( x_0 \) if there exists a linear functional \( L \), which depends on \( x_0 \), such that \( f(x_0 + h) = f(x_0) + Lh + o(|h|) \) in a neighborhood of \( x_0 \). If such functional were to exist, then

\[
Lv = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{|h|}
\]

where \( v = h/|h| \) is a unit vector. Since \( f \) is a radial function, namely it depends upon \( r = \sqrt{x^2 + y^2} \) only, it is immediate to show that the previous limit exists and is equal to zero. Thus, \( f \) is differentiable at the origin and \( L = 0 \).

ii. True. Since \( f \) is differentiable, then its differential can be expressed via the gradient.

iii. True. Indeed:

\[
\frac{∂f}{∂x}(0, 0) = \lim_{h \to 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h \to 0} \frac{h^2 \sin(1/|h|)}{h} = 0
\]

and analogously for the other partial derivative.

iv. False. Indeed, away from the origin:

\[
\frac{∂f}{∂x} = 2x \sin(1/\sqrt{x^2 + y^2}) - \frac{x \cos(1/\sqrt{x^2 + y^2})}{\sqrt{x^2 + y^2}}
\]
\[
\frac{∂f}{∂y} = 2y \sin(1/\sqrt{x^2 + y^2}) - \frac{y \cos(1/\sqrt{x^2 + y^2})}{\sqrt{x^2 + y^2}}
\]

The limits of those functions, as we approach the origin, clearly do not exist.

Notice that \( f \) is an example of a function that, although being differentiable in a neighborhood of the origin, does not have continuous partial derivatives at the origin.