1. Calculate
$$\int_{\mathbb{R}^2} \frac{1}{(1 + x^2 + (y-x)^2)^2} \, dx \, dy.$$  

Solution. By invariance under translations
$$\int_{\mathbb{R}^2} \frac{1}{(1 + x^2 + (y-x)^2)^2} \, dx \, dy = \int_{\mathbb{R}^2} \frac{1}{(1 + x^2 + y^2)^2} \, dx \, dy.$$
Using polar coordinates $x = r \cos \theta, \quad y = r \sin \theta$ we get
$$\int_{\mathbb{R}^2} \frac{1}{(1 + x^2 + (y-x)^2)^2} \, dx \, dy = 2\pi \int_{0}^{\infty} \frac{r}{(1 + r^2)^2} \, dr = \pi \int_{0}^{\infty} \frac{d}{dr} \frac{1}{1 + r^2} \, dr = \pi.$$

2. Calculate
$$\int_{\mathbb{R}^3} e^{-x^2-2y^2-3z^2} \, dx \, dy \, dz.$$  

Solution.
$$\int_{\mathbb{R}^3} e^{-x^2-2y^2-3z^2} \, dx \, dy \, dz = \int_{\mathbb{R}} e^{-x^2} \, dx \int_{\mathbb{R}} e^{-2y^2} \, dy \int_{\mathbb{R}} e^{-3z^2} \, dz$$
$$= \frac{1}{\sqrt{2}} \int_{\mathbb{R}} e^{-x^2} \, dx \frac{1}{2} \int_{\mathbb{R}} e^{-y^2} \, dy \frac{1}{\sqrt{6}} \int_{\mathbb{R}} e^{-z^2} \, dz$$
$$= \frac{(2\pi)^{3/2}}{4\sqrt{3}} \frac{\pi^{3/2}}{\sqrt{6}} \frac{\pi^{3/2}}{6} = \frac{\pi^{3/2}}{3}.$$

3. Let
$$E = \{(x, y, z) \in \mathbb{R}^3 : x \in [0, 1], y^2 + z^2 \leq x^2\}.$$  
Describe $E$ and give $|E| = \text{Vol}(E)$.

Solution. The set $E$ represents a cone around the axis $x$. The summit is $(0, 0, 0)$.
$$|E| = \pi \int_{0}^{1} x^2 \, dx = \frac{\pi}{3}.$$

4. For the differential equation:
$$\frac{dy}{dx} = 9x^2 \, y$$  

find the general solution.
(a) $y(x) = Ae^{3x^3}$  
(b) $y(x) = Ae^{3x^4}$  
(c) $y(x) = Ae^{x^2}$  
(d) $y(x) = Ae^{x^3}$
Solution. The correct answer is (a). By separating the variables, the equation becomes:

\[ \frac{dy}{y} = 9x^2\,dx \]

and after integrating both sides we get:

\[ \ln y = 3x^3 + C \]

and so \( y = Ae^{3x^3} \).

5. Suppose that \( y_0 \) satisfies:

\[ (x^2 + 9)\frac{dy}{dx} = xy, \quad y(0) = 3. \]

Find the value of \( y_0(9) \).

(a) \( y_0(9) = 4\sqrt{10} \)
(b) \( y_0(9) = 3\sqrt{10} \)
(c) \( y_0(9) = 40 \)
(d) \( y_0(9) = 3\sqrt{17} \)

Solution. The correct answer is (b). The general solution to the equation can be found by separation of variables:

\[ \frac{dy}{y} = \frac{dx}{x^2 + 9} \]

and after integrating both sides we get \( y_0 = A(x^2 + 9)^{1/2} \). If we require \( y_0(0) = 3 \) then \( A = 1 \) and hence \( y_0(9) = 3\sqrt{10} \).

6. Suppose that \( y_0 \) satisfies:

\[ (x + 3)\frac{dy}{dx} = y - 1, \quad y(1) = 2. \]

Find the value of \( y_0(4) \).

(a) \( y_0(4) = \frac{7}{2} \)
(b) \( y_0(4) = -1 \)
(c) \( y_0(4) = 3 \)
(d) \( y_0(4) = 11/4 \)

Solution. The correct answer is (d). The general solution to the equation can be found by separation of variables:

\[ \frac{dy}{y - 1} = \frac{dx}{x + 3} \]

and after integrating both sides we get \( y_0 = 1 + A(x + 3) \). If we require \( y_0(1) = 2 \) then \( A = 1/4 \) and hence \( y_0(4) = 11/4 \).

7. For the differential equation:

\[ \frac{dy}{dx} = \frac{e^{5x}}{6y^2} \]

find the general solution.

(a) \( y(x) = \pm \sqrt{e^{3x}/5 + C} \)
(b) \( y(x) = \pm \sqrt{e^{3x}/5 + C} \)
(c) \( y(x) = \pm \sqrt{e^{3x}/5 + C} \)
(d) \( y(x) = \pm \sqrt{e^{3x}/5 + C} \)
Solution. The correct answer is (c). By separating the variables, the equation becomes:

$$6y^5 dy = e^{5x} dx$$

and after integrating both sides we get:

$$y^6 = e^{5x}/5 + C$$

and so \( y(x) = \pm \sqrt[6]{e^{5x}/5 + C} \).

8. Find the general solution of the equation:

$$2y dy/dx = 9x.$$

(a) \( y = \pm \sqrt{\frac{2}{3} x^2 + C} \)

(b) \( y = \pm \sqrt{\frac{2}{3} x^2 + C} \)

(c) \( y = \pm \sqrt{\frac{2}{3} x^2} \)

(d) \( y = \pm \sqrt{\frac{2}{3} x^2 + C} \)

Solution. The correct answer is (a). If we separate the variables we get:

$$2y dy = 9x dx$$

which integrates to \( 2y^2 = 9x^2 + C \) from which we get (a).

9. The solution \( y(x) \) of the differential equation \((x^2 + 9)y' + xy - xy^2 = 0\) for \( x \in \mathbb{R} \) with the initial condition \( y(0) = 1/4 \) also satisfies:

(a) \( y(4) = 1/6 \)

(b) \( y(4) = -1/4 \)

(c) \( y(4) = 6 \)

(d) \( y(4) = 1 \)

Solution. The correct answer is (b). If we separate the variables we get:

$$\frac{dy}{y^2 - y} = \frac{x}{x^2 + 9} dx$$

We have

$$\int \frac{dy}{y^2 - y} = \int \frac{1}{y - 1} - \frac{1}{y} dy = \ln\left(\frac{y - 1}{y}\right) + C_1, \quad \frac{y - 1}{y} > 0$$

and

$$\int \frac{x}{x^2 + 9} dx = \frac{1}{2} \ln(x^2 + 9) + C_2$$

If we put everything together we get

$$\ln\left(\frac{y - 1}{y}\right) = \frac{1}{2} \ln(x^2 + 9) + C \implies \left(\frac{y - 1}{y}\right)^2 = A(x^2 + 9)$$

If we use the initial condition \( y(0) = 1/4 \) we get that \( A = 1 \) and finally for we can compute \( y(4) \),

$$\left(\frac{y - 1}{y}\right)^2 = 25 \implies y(4) = 1/6 \text{ or } y(4) = -1/4$$

where \( y = -1/4 \) is the acceptable solution.

10. Find the general solution of the following equations

(a) \( y' - \frac{3y}{x+1} = (x+1)^4 \)

(b) \( \cos(x)y' + \sin(x)y = 2\cos^3(x)\sin(x) - 1 \)
Solution.

(a) The differential equation is of the form \( y' + P(x)y = Q(x) \). We first find the integrating factor

\[
I = e^{\int P(x) \, dx} = e^{\int \frac{-3}{x+1} \, dx} = e^{-3 \ln(x+1)} = e^{\ln(x+1)^{-3}} = \frac{1}{(x+1)^3}
\]

We multiply both sides of differential equation with \( I \) to get

\[
\frac{1}{(x+1)^3}y' - \frac{3y}{(x+1)^4} = (x+1)
\]

by integrating both sides we get

\[
\frac{y}{(x+1)^3} = \frac{1}{2} x^2 + x + C
\]

So the general solution is

\[
y = (x+1)^3 \left( \frac{1}{2} x^2 + x + C \right)
\]

(b) We first write the differential equations in the form of \( y' + P(x)y = Q(x) \):

\[ y' + \frac{\sin(x)}{\cos(x)} y = 2 \cos^2(x) \sin(x) - \frac{1}{\cos(x)} \]

Now we find the integral factor

\[
I = e^{\int P(x) \, dx} = e^{\int \frac{\sin(x)}{\cos(x)} \, dx} = e^{- \ln |\cos(x)|} = \frac{1}{\cos(x)}
\]

Now we multiply both sides of the differential equation with \( I \)

\[
\frac{y'}{\cos(x)} + \frac{\sin(x)}{\cos^2(x)} = 2 \sin(x) \cos(x) - \frac{1}{\cos^2(x)}
\]

Taking the integral of both sides yields

\[
\frac{y}{\cos(x)} = -\frac{1}{2} \cos(x) - \tan(x) + C
\]

So the general solution is

\[
y = -\frac{1}{2} \cos(x) \cos(2x) - \sin(x) + C \cos(x)
\]

11. For each of the following differential equations check if the solution exists and is unique.

(a) \( y' = 1 + y^2 \), \( y(0) = 0 \)

(b) \( y' = \frac{2y}{x} \), \( y(a) = b \)

solve the differential equation (b) and sketch the family of solutions for some initial conditions \( y(a) = b \). What happens when \( a = 0 \) or \( b = 0? \) Compare this with the existence-uniqueness theorem.

Solution.

(a) Let \( F(x, y) = 1 + y^2 \). Then both \( F(x, y) \) and \( \frac{\partial}{\partial y} F(x, y) = 2y \) are defined and continuous at all points \((x, y)\), so by the theorem we can conclude that a solution exists in some open interval centered at 0, and is unique in some (possibly smaller) interval centered at 0.
(b) In this example, \( F(x, y) = \frac{2y}{x} \) and \( \frac{\partial}{\partial y} F(x, y) = \frac{2}{x} \). Both of these functions are defined for all \( x \neq 0 \) so the existence-uniqueness theorem tells us that for each \( a \neq 0 \) there exists a unique solution defined in an open interval around \( a \). By separating variables and integrating, we derive solutions to this equation of the form

\[
y = Cx^2
\]

for any constant \( C \). Notice that all of these solutions pass through the point \((0, 0)\), and that none of them pass through any point \((0, b)\) with \( b \neq 0 \). So the initial value problem

\[
y' = \frac{2y}{x}, \quad y(0) = 0
\]

has infinitely many solutions, but the initial value problem

\[
y' = \frac{2y}{x}, \quad y(0) = b, \quad b \neq 0
\]

has no solutions.