Solutions to Exercise Session, March 14, 2016

1. **Level curves.** Find the equation of the level curve of the function $f(x, y)$ that passes through the given point.

   (a) $f(x, y) = 16 - x^2 - y^2$, $(2\sqrt{2}, \sqrt{2})$

   **Solution.** $f(2\sqrt{2}, \sqrt{2}) = 6$, so the level curve have the equation $x^2 + y^2 = 10$.

   (b) $f(x, y) = \sqrt{x^2 - 1}$, $(1, 0)$

   **Solution.** $f(1, 0) = 0$, so the level curve have the equation $x^2 = 1$ which consist of two lines $x = 1$ and $x = -1$.

   (c) $f(x, y) = \int_x^y \frac{d\theta}{\sqrt{1 - \theta^2}}$, $(0, 1)$

   **Solution.** We have that $\int_x^y \frac{d\theta}{\sqrt{1 - \theta^2}} = \arcsin y - \arcsin x$ where $-1 \leq x \leq 1$ and $-1 \leq y \leq 1$. We also have that $f(0, 1) = \pi/2$. In order for $\arcsin y - \arcsin x$ to be equal to $\pi/2$ we must have $0 \leq \arcsin y \leq \pi/2$ and $-\pi/2 \leq \arcsin x \leq 0$ meaning $0 \leq y \leq 1$ and $-1 \leq x \leq 0$. So the equation of the curve is given by

   $\arcsin y - \arcsin x = \frac{\pi}{2} \implies y = \sin(\frac{\pi}{2} - \arcsin x) \implies y = \sqrt{1 - x^2}$, $x \leq 0$.

2. **Continuous functions.**

   (a) Let $a, b \in \mathbb{R}^n$, $a, b \neq 0$. Show that the function $f(x) = \langle a, x \rangle \cdot \langle b, x \rangle$ is continuous for all $x \in \mathbb{R}^n$.

   **Solution.** It is the product of two continuous functions (linear forms), so we get the continuity of $f$. Indeed, let $x, x_j \in \mathbb{R}^n$ s.t. $\lim_{j \to \infty} x_j = x$. Then, by continuity of linear forms

   $\lim_{j \to \infty} \langle a, x_j \rangle = \langle a, x \rangle$, $\lim_{j \to \infty} \langle b, x_j \rangle = \langle b, x \rangle$,

   hence $\lim_{j \to \infty} f(x_j) = f(x)$ since it is the product of two convergent numerical sequences.

   (b) For $A \in M_{n,n}(\mathbb{R})$ let $b : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a bilinear form given by $b(x, y) = \langle x, Ay \rangle$. Show that $b$ is continuous for all $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^{2n}$. 


Solution-1. For all sequences of vectors \( \left( \begin{array}{c} x_j \\ y_j \end{array} \right) \in \mathbb{R}^{2n} \) that converge to \( \left( \begin{array}{c} x \\ y \end{array} \right) \in \mathbb{R}^{2n} \), we have \( \lim_{j \to \infty} x_j = x \) and \( \lim_{j \to \infty} y_j = y \) in \( \mathbb{R}^n \), i.e. \( \lim_{j \to \infty} \| x_j - x \|_2 = 0 \), \( \lim_{j \to \infty} \| y_j - y \|_2 = 0 \) (with the Euclidean norm in \( \mathbb{R}^n \)). In particular, \( \| x_j \|_2, \| y_j \|_2 \) are bounded. By bilinearity of \( b \):

\[
b(x_j, y_j) - b(x, y) = b(x_j - x, y_j) + b(x, y_j - y).
\]

By Cauchy-Schwarz’s inequality and the inequality \( \| Ax \|_2 \leq \| A \|_2 \| x \|_2 \) (see lecture, ch.1.6.1, p.14) we get:

\[
|b(x_j, y_j) - b(x, y)| \leq \| A \|_2 \| y_j \|_2 \| x_j - x \|_2 + \| A \|_2 \| x \|_2 \| y_j - y \|_2 \to 0
\]

hence the result.

Solution-2. The game \( \epsilon - \delta \). We have to show that for all \( \epsilon < 0 \) there exists \( \delta < 0 \) such that for all vectors \( \left( \begin{array}{c} h \\ k \end{array} \right) \in \mathbb{R}^{2n} \) of Euclidean norm smaller than \( \delta \) we get

\[
|b(x + h, y + k) - b(x, y)| < \epsilon.
\]

Note that for the Euclidean norm in \( \mathbb{R}^{2n} \):

\[
\left\| \left( \begin{array}{c} h \\ k \end{array} \right) \right\|_2^2 = \| h \|^2_2 + \| k \|^2_2
\]

with the norms on the right taken in \( \mathbb{R}^n \). By estimation of the solution 1 above, for all \( x, y, h, k \in \mathbb{R}^n \):

\[
|b(x + h, y + k) - b(x, y)| \leq \| A \|_2 \| y + k \|_2 \| h \|_2 + \| A \|_2 \| x \|_2 \| k \|_2.
\]

We can assume that \( \| y + k \|_2 < C, \| x \|_2 < C \) for a constant \( C > 0 \). Then,

\[
|b(x + h, y + k) - b(x, y)| \leq C\| A \|_2 (\| h \|_2 + \| k \|_2) \leq C\| A \|_2 \sqrt{2\| h \|^2_2 + 2\| k \|^2_2}
\]

by the inequality \( a + b \leq \sqrt{2a^2 + 2b^2} \) for all \( a, b \geq 0 \). We choose \( \delta = \frac{\epsilon}{\sqrt{2C\| A \|_2}} \).

(c) Let \( g : \mathbb{R} \to \mathbb{R} \) be a continuous function. Show that \( f : \mathbb{R}^n \to \mathbb{R} \) given by \( f(x) = \sum_{k=1}^{n} g(x_k) \), where \( x_k \) denotes the \( k^{th} \) component of the vector \( x \), \( x_k = \langle e_k, x \rangle \), is a continuous function for all \( x \in \mathbb{R}^n \).

Solution. Either by sequences or by

\[
f(x + h) - f(x) = \sum_{k=1}^{n} g(x_k + h_k) - g(x_k)
\]

and \( |h_k| \leq \| h \|_2 \), applying the continuity of \( g \):

\[
\lim_{h \to 0} f(x + h) - f(x) = \sum_{k=1}^{n} \lim_{h_k \to 0} g(x_k + h_k) - g(x_k) = 0.
\]

Alternatively we can argue that the functions \( h_k : \mathbb{R}^n \to \mathbb{R} \) defined by \( h_k(x) = g(\langle e_k, x \rangle) \) are continuous on \( \mathbb{R}^n \) (it is the composition of a continuous function with a continuous linear form- see exercise under) and \( f \) is a finite sum of continuous functions.

(d) Let \( g : \mathbb{R} \to \mathbb{R}, \; h : \mathbb{R}^n \to \mathbb{R} \) be continuous functions. Show that \( f : \mathbb{R}^n \to \mathbb{R} \) given by \( f(x) = g(h(x)) \) is a continuous function for all \( x \in \mathbb{R}^n \).
Solution. Let \( x, x_j \in \mathbb{R}^n \) s.t. \( \lim_{j \to \infty} x_j = x \). Then, by continuity of \( h \), the numerical sequence \( a_j := (h(x_j)) \) is convergent and has for limit \( a := h(x) \). By the continuity of \( g \): \( \lim_{j \to \infty} g(a_j) = g(a) \), i.e.

\[
\lim_{j \to \infty} f(x_j) = f(x).
\]

3. Limits of real functions.

(a) Calculate

\[
\lim_{(x,y) \to (0,0)} \frac{x^2 - y^2}{x^2 + y^2}
\]

Solution. The limit doesn’t exist since \( f(0,y) = -1 \) for \( y \neq 0 \) and \( f(x,0) = 1 \) if \( x \neq 0 \).

(b) Calculate

\[
\lim_{(x,y) \to (0,0)} xy \frac{x^2 - y^2}{x^2 + y^2}
\]

Solution. Note that

\[
|xy \frac{x^2 - y^2}{x^2 + y^2}| \leq |xy| \leq x^2 + y^2
\]

Hence

\[
\lim_{(x,y) \to (0,0)} xy \frac{x^2 - y^2}{x^2 + y^2} = 0
\]

(c) Calculate

\[
\lim_{(x,y) \to (0,0)} e^{-\frac{1}{\sqrt{x^2+y^2}}}
\]

Solution. We use polar coordinates. We substitute \( x = r \cos \theta \) and \( y = r \sin \theta \) and investigate the limit of resulting expression as \( r \to 0 \).

\[
\lim_{(x,y) \to (0,0)} e^{-\frac{1}{\sqrt{x^2+y^2}}} = \lim_{r \to 0} e^{-\frac{1}{r^2}} = 0
\]

(d) Show that the function

\[
f(x, y) = \frac{2x^2y}{x^4 + y^2}
\]

has no limit as \( (x, y) \) approaches \( (0,0) \). In particular show the value of the limit take varies between \(-1\) and \(1\) along curves \( y = kx^2 \).

Solution. We take the limit along the curve \( y = kx \). If \( x \neq 0 \)

\[
f(x, y)\bigg|_{y=kx^2} = \frac{2kx^4}{(1+k^2)x^4} = \frac{2k}{1+k^2}
\]

So

\[
\lim_{y \to kx^2} f(x, y) = \lim_{(x,y) \to (0,0)} f(x, y)\bigg|_{y=kx^2} = \frac{2k}{1+k^2}
\]

This limit varies with the path of approach. Now take \( k = \tan \theta \) then

\[
\frac{2k}{1+k^2} = \frac{2\tan \theta}{1+\tan^2 \theta} = \sin 2\theta
\]

And \( \sin 2\theta \) varies between \(-1\) and \(1\).
Let \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \) be the continuous function defined by
\[
f(x, y) = \begin{cases} 
\frac{\sin(xy)}{xy} & \text{if } xy \neq 0, \\
1 & \text{if } xy = 0.
\end{cases}
\]

Show that \( f \) is partially differentiable and give its partial derivatives.

**Solution.** If \( xy \neq 0 \), then
\[
D_x \frac{\sin(xy)}{xy} = \frac{x y^2 \cos(xy) - y \sin(xy)}{x^2 y^2}
\]
\[
D_y \frac{\sin(xy)}{xy} = \frac{x^2 y \cos(xy) - x \sin(xy)}{x^2 y^2}
\]

If \( xy = 0 \), there are three cases: \( x = 0, y \neq 0 \) or \( x \neq 0, y = 0 \) or also \( x = 0, y = 0 \). For example, for the first case:
\[
D_x f(0, y) = \lim_{h \to 0} \frac{f(h, y) - f(0, y)}{h} = 0
\]
and
\[
D_y f(0, y) = \lim_{h \to 0} \frac{f(0, y + h) - f(0, y)}{h} = 0
\]

4. **Continuity.** Study continuity of following functions as a function of \( \alpha > 0 \).

(a)
\[
f(x, y) = \begin{cases} 
x y (x^2 + y^2)^{\alpha} & \text{if } (x, y) \neq (0, 0) \\
0 & \text{if } (x, y) = (0, 0)
\end{cases}
\]

**Solution.** For \( (x, y) \neq (0, 0) \) the denominator is non-zero and \( f \) is a combination of continuous functions. Therefore for all \( \alpha > 0 \), \( f(x, y) \) is continuous \( \forall (x, y) \neq (0, 0) \). We check the continuity at \( (x, y) = (0, 0) \). Using polar coordinates \( x = r \cos \theta \) and \( y = r \sin \theta \) we have
\[
\lim_{(x,y) \to (0,0)} \frac{r^{2\alpha} \cos^{2\alpha} \theta}{r^2} = \lim_{r \to 0} \frac{r^{2\alpha} \cos^{2\alpha} \theta}{r^2}
\]
The value of the limit depends on \( \alpha \):
- case \( \alpha > 1 \): The limit is 0 because \( |r^{2\alpha} \cos^{2\alpha} \theta| \leq |r^{2\alpha}| \to 0 \)
- case \( \alpha = 1 \): The value of the limit is 1 \cdot \cos \theta
- case \( 0 < \alpha < 1 \): The limit is \( +\infty \) if \( \cos \theta \neq 0 \) and the limit is 0 if \( \cos \theta = 0 \).

So \( f \) is continuous on \( \mathbb{R}^2 \) if \( \alpha > 1 \) and is continuous on \( \mathbb{R}^2 \setminus (0, 0) \) when \( 0 < \alpha \leq 1 \)

(b)
\[
f(x, y) = \begin{cases} 
x y (x^2 + y^2)^{\alpha} & \text{if } (x, y) \neq (0, 0) \\
0 & \text{if } (x, y) = (0, 0)
\end{cases}
\]

**Solution.** For \( (x, y) \neq (0, 0) \) the denominator is non-zero and \( f \) is a combination of continuous functions. Therefore for all \( \alpha > 0 \), \( f(x, y) \) is continuous \( \forall (x, y) \neq (0, 0) \). We check the continuity at \( (x, y) = (0, 0) \). Using polar coordinates \( x = r \cos \theta \) and \( y = r \sin \theta \) we have
\[
\lim_{(x,y) \to (0,0)} \frac{x y (x^2 + y^2)^{\alpha}}{(x^2 + y^2)^{\alpha}} = \lim_{r \to 0} r^{2(1-\alpha)} \cos \theta \sin \theta
\]
The value of the limit depends on \( \alpha \):
5. Partial derivatives.

(a) Let \(a, b \in \mathbb{R}^n, a, b \neq 0\). Show that the function \(f(x) = \langle a, x \rangle \cdot \langle b, x \rangle\) is partially differentiable for all \(x \in \mathbb{R}^n\) and give its gradient.

**Solution-1.** \(f = g \cdot h\) is the product of two partially differentiable functions \(g(x) = \langle a, x \rangle\) and \(h(x) = \langle b, x \rangle\). By the product rule:

\[
\nabla f(x) = \nabla (gh)(x) = h(x)\nabla g(x) + g(x)\nabla h(x).
\]

It follows that \(\nabla g(x) = a, \nabla h(x) = b\):

\[
\nabla f(x) = \langle b, x \rangle a + \langle a, x \rangle b.
\]

**Solution-2.** For all \(k = 1, \ldots, n, t \in \mathbb{R}\) and \(x \in \mathbb{R}^n\):

\[
f(x + te_k) - f(x) = \langle a, e_k \rangle \langle b, x + te_k \rangle + t\langle a, x \rangle \langle b, e_k \rangle
\]

hence

\[
t^{-1}(f(x + te_k) - f(x)) = a_k\langle b, x \rangle + b_k\langle a, x \rangle + t a_k b_k.
\]

By letting \(t\) go to zero, we get the result.

(b) For \(A \in M_{n,n}(\mathbb{R})\), let \(b : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}\) be the bilinear form given by \(b(x, y) = \langle x, Ay \rangle\). Show that \(b\) is partially differentiable for all \(\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^{2n}\) and give its gradient.

**Solution.** For the partial derivatives of \(x_k\), the argument \(y\) is constant, hence it is the study of the linear form \(x \mapsto \langle x, Ay \rangle\). We find \(\nabla_x b(x, y) = Ay\). For the partial derivatives of \(y_k\) the argument \(x\) is constant, hence it is the study of the linear form \(y \mapsto \langle A^T x, y \rangle\) (we have to put the matrix in the constant argument). We find \(\nabla_y b(x, y) = A^T x\). The gradient of \(b\) is the vector in \(\mathbb{R}^{2n}\) given by

\[
\nabla b(x, y) = \nabla_{x,y} b(x, y) = \begin{pmatrix} Ay \\ A^T x \end{pmatrix}.
\]

(c) Let \(g : \mathbb{R} \rightarrow \mathbb{R}\) be a differentiable function. Show that \(f : \mathbb{R}^n \rightarrow \mathbb{R}\) given by \(f(x) = \sum_{k=1}^{n} g(x_k)\), where \(x_k\) denotes the \(k^{th}\) component of the vector \(x\), \(x_k = \langle e_k, x \rangle\), is a partially differentiable function for all \(x \in \mathbb{R}^n\). Give its gradient.

**Solution.** By the definition of partial derivatives:

\[
\frac{\partial f(x)}{\partial x_j} = \lim_{h \rightarrow 0} \frac{f(x + he_j) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{g(x_j + h) - g(x_j)}{h} = g'(x_j)
\]

for all \(x \in \mathbb{R}^n\) and \(g'\) denotes the derivative function of \(g\). Hence,

\[
\nabla f(x) = \sum_{k=1}^{n} g'(x_k)e_k.
\]

(d) Let \(g : \mathbb{R} \rightarrow \mathbb{R}\) be differentiable for all \(t \in \mathbb{R}\), \(h : \mathbb{R}^n \rightarrow \mathbb{R}\) partially differentiable for all \(x \in \mathbb{R}^n\). Show that \(f : \mathbb{R}^n \rightarrow \mathbb{R}\) given by \(f(x) = g(h(x))\) is a partially differentiable function for all \(x \in \mathbb{R}^n\). Give its gradient.
Solution. The function \( \rho(t) := h(x+te_k) \) is differentiable in \( t = 0 \) and \( \rho'(0) = \frac{\partial h(x)}{\partial x_k} \).

The composite function \( g(\rho(t)) \) is differentiable at \( t = 0 \) and

\[
\frac{\partial f(x)}{\partial x_k} = \left. \frac{d}{dt} \right|_{t=0} g(\rho(t)) = g'(\rho(0))\rho'(0) = g'(h(x))\frac{\partial h(x)}{\partial x_k},
\]

hence \( \nabla f(x) = g'(h(x))\nabla h(x) \).

6. Let \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \) be the function defined by

\[
f(x, y) = \begin{cases} 
(x^2 + y^2) \sin \left( \frac{1}{\sqrt{x^2+y^2}} \right) & \text{if } (x, y) \neq (0, 0), \\
0 & \text{if } (x, y) = (0, 0).
\end{cases}
\]

Show that \( f \) is differentiable at \((0, 0)\) but is not of class \( C^1 \) at this point.

Solution. Let \( r = \sqrt{x^2+y^2}, \ r \geq 0 \). \( f \) is differentiable at \((0, 0)\) and \( d_0f(x, y) = 0 \) since

\[
\lim_{r \to 0^+} \frac{f(x, y) - f(0, 0)}{r} = r \sin \left( \frac{1}{r} \right) = 0
\]

If \((x, y) \neq (0, 0)\) the function \( f \) is partially differentiable (even differentiable) and noting that \( f \) is radially symmetric:

\[
\frac{\partial f(x, y)}{\partial x} = \frac{x}{r} \left( 2r \sin r^{-1} - \cos r^{-1} \right),
\]

\[
\frac{\partial f(x, y)}{\partial y} = \frac{y}{r} \left( 2r \sin r^{-1} - \cos r^{-1} \right)
\]

These functions don’t have any limits when \((x, y) \to (0, 0)\) (because of \( \cos r^{-1} \)).

7. For \( x \in \mathbb{R} \) and \( t > 0 \) we consider the function \( f(x, t) \) defined by

\[
f(x, t) = \frac{1}{\sqrt{4\pi t}} \exp \left( -\frac{x^2}{4t} \right).
\]

(a) Show that \( f \) verifies the heat equation, i.e.

\[
\frac{\partial f}{\partial t}(x, t) - \frac{\partial^2 f}{\partial x^2}(x, t) = 0
\]

Solution.

\[
\frac{\partial f}{\partial t}(x, t) = -\frac{x}{2t} f(x, t)
\]

and

\[
\frac{\partial^2 f}{\partial x^2}(x, t) = -\frac{1}{2t} f(x, t) - \frac{x}{2t} \frac{\partial f}{\partial x}(x, t) = \left( -\frac{1}{2t} + \frac{x^2}{4t^2} \right) f(x, t) = \frac{\partial f}{\partial t}(x, t)
\]

(b) Calculate

\[
\int_{\mathbb{R}} f(x, t) \, dx
\]
Solution. By the change of variable $y = x / \sqrt{2t}$ i.e. $dx/dy = \sqrt{2t}$, we get the Gauss integral:

$$\int_{-\infty}^{\infty} f(x, t) \, dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \, dy = 1$$

(c) Let $g(x, y, t)$ given by $g(x, y, t) = f(x, t) f(y, t)$. Calculate

$$\frac{\partial g}{\partial t}(x, y, t) - \frac{\partial^2 g}{\partial x^2}(x, y, t) - \frac{\partial^2 g}{\partial y^2}(x, y, t).$$

Remark: $\frac{\partial^2}{\partial x^2} = D_{xx}$ etc.

Solution. By the product rule and the result in (a) we get

$$\frac{\partial g}{\partial t}(x, y, t) - \frac{\partial^2 g}{\partial x^2}(x, y, t) - \frac{\partial^2 g}{\partial y^2}(x, y, t) =$$

$$\frac{\partial f}{\partial t}(x, t) f(y, t) + f(x, t) \frac{\partial f}{\partial t}(y, t) - \frac{\partial^2 f}{\partial x^2}(x, t) f(y, t) - f(x, t) \frac{\partial^2 f}{\partial y^2}(y, t) =$$

$$f(x, t) \left( \frac{\partial f}{\partial t}(y, t) - \frac{\partial^2 f}{\partial y^2}(y, t) \right) + f(y, t) \left( \frac{\partial f}{\partial t}(x, t) - \frac{\partial^2 f}{\partial x^2}(x, t) \right) = 0.$$

8. True or False.

(a) A continuous function is partially differentiable. □ True □ False

Solution. False, for example take $f(x) = \|x\|_2$ which is a continuous function and not differentiable.

(b) If all the directional derivatives of $f$ exist, then all the partial derivatives also exist. □ True □ False

Solution. True, Just take the directions to be the basis of the space.

(c) If all the partial derivatives of $f$ exist, then all the directional derivative also exist. □ True □ False

Solution. True, write any vector $v$ as a linear combination of basis vectors then the statement follows immediately.

(d) If all the partial derivatives of $f$ exist, then $f$ is continuous. □ True □ False

Solution. False, take the following function for example

$$f(x, y) = \begin{cases} 0, & xy \neq 0 \\ 1, & xy = 0 \end{cases}$$

all partial derivatives exist but is not continuous.