Exercise Session, April 11, 2016

1. **Implicit functions I.** Show that the equation

\[ \ln x + e^y x^y = 1 \]

defined in the neighborhood of the point 1 is an implicit function \( y = g(x) \) such that \( g(1) = 0 \). Give the equation of the tangent to the curve \( y = g(x) \) at 1.

2. **Implicit functions II.** Show that the equation

\[ \cos(x^2 + y) + \sin(x + y) + e^{x^3 y} = 2 \]

defined in the neighborhood of the point 0 is an implicit function \( y = g(x) \) such that \( g(0) = \pi/2 \). Show that the function \( g \) has a local maximum at 0.

3. **Implicit functions III.** Show that the equation

\[ x^5 + xyz + y^3 + 3xz^4 = 2 \]

defined in the neighborhood of the point \((1, -1)\) is an implicit function \( z = g(x, y) \) such that \( g(1, -1) = 1 \). Give the equation of the plane tangent to the surface \( z = g(x, y) \) in \((1, -1)\).

4. **Quadratic form.** Let \( A \in M_{n,n}(\mathbb{R}) \) be a positive-definite, symmetric matrix. Let \( \mathbf{v} \in \mathbb{R}^n \). Show that the function \( f(x) \) defined by

\[ f(x) = \frac{1}{2} \langle A \mathbf{x}, \mathbf{x} \rangle - \langle \mathbf{v}, \mathbf{x} \rangle \]

has a unique stationary point at \( \mathbf{a} = A^{-1} \mathbf{v} \). Then show that \( f(x) - f(a) > 0 \) for all \( x \neq a \).

5. Study the nature of the stationary points of the function \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \) given by

\[ f(x, y) = (1 - x^2) \sin y. \]

6. Let \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \) be defined by

\[ f(x, y) = (x - y)^3 + 4x^2 - 3x + 3y. \]

(a) Give the stationary points of \( f \) and study their nature. Calculate \( f \) at these points.

(b) Let \( T \) be the domain given by:

\[ T = \{(x, y) \in \mathbb{R} : y \geq 0, y \leq x \leq 4 - y\}. \]

Give the minimum and the maximum of \( f \) on \( T \). In particular,

i. Show that \( T \) is bounded.

ii. Show that \( \partial T \subset T \) and conclude that \( T \) is closed.

iii. Show that \( T \) is a triangle and give its summits.

iv. Explain why \( f \) has its maximum and minimum on \( T \).

v. Give \( f \) on the boundary of \( T \), i.e. \( f|_{\partial T} \) and then study \( f|_{\partial T} \).
vi. Give the minimum and the maximum of $f$ on $T$.

7. Calculate the extrema of the function

$$f(x, y) = x^4 + y^4$$

under the constrain $g(x, y) = xy - 1 = 0$.

(a) Find the extrema directly (by replacing the constrain $g$ in $f$).

(b) Find the extrema using Lagrange multiplier.

8. Compute the extrema of the function $f(x, y) = x^2 + y^2$ under the constraint $g(x, y) = (x - 1)^2 + (y - 1)^2 - 4$.

9. The atmospheric pressure in a region of space near the origin is given by the formula

$$P = 30 + (x + 1)(y + 2)e^z.$$  

Approximately where is the point closest to the origin at which the pressure is 31.1.  

(Hint: linearize the equation around the origin. Then find the point closest to the origin that satisfy the linearized equation.)