Exercise Session, April 18, 2015

1. **Jacobian matrix.** Find the Jacobian matrix of the following maps:

   (a) Let \( u: \mathbb{R}^2 \rightarrow \mathbb{R}^3 \) be defined as:
   \[
   u(x, y) = \begin{pmatrix}
   -y \\
   x \\
   x + y
   \end{pmatrix}
   \]
   (b) Let \( v: \mathbb{R}^2 \rightarrow \mathbb{R}^3 \) and \( w: \mathbb{R}^3 \rightarrow \mathbb{R}^2 \) be defined as:
   \[
   v(x, y) = \begin{pmatrix}
   -y \\
   x \\
   xy
   \end{pmatrix}
   \]
   \[
   w(x, y, z) = \begin{pmatrix}
   x^2 + y^2 - 2z \\
   x^2 + y^2 + 2z
   \end{pmatrix}
   \]
   Find the Jacobian matrix of \( w \circ v \) by (i) computing the composition and then its Jacobian matrix; (ii) using the chain rule.

   (c) Let \( v: \mathbb{R}^3 \rightarrow \mathbb{R}^2 \) and \( w: \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) be given by
   \[
   v(x, y, z) = \begin{pmatrix}
   e^{y+2z} \\
   x^2 + yz
   \end{pmatrix}
   \]
   \[
   w(x, y) = \begin{pmatrix}
   \cos x \\
   \sin y
   \end{pmatrix}
   \]
   Find the Jacobian matrix of \( w \circ v \) by (i) computing the composition and then its Jacobian matrix; (ii) using the chain rule.

2. **Jacobian matrix.** Let \( v: \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) be defined as
   \[
   v(r, \theta, \phi) = (r \sin \theta \cos \phi, r \sin \theta \sin \phi, r \cos \theta).
   \]
   Find the Jacobian matrix \( J_v \) and the Jacobian, i.e., the determinant \( \det J_v \).

3. Find the Jacobian matrices of the following maps:
   \[
   v: \mathbb{R}^2 \rightarrow \mathbb{R}^3, \quad v(x, y) = \begin{pmatrix}
   \frac{2x}{1+x^2+y^2} \\
   \frac{2y}{1+x^2+y^2} \\
   \frac{1-x^2-y^2}{1+x^2+y^2}
   \end{pmatrix}
   \]
   \[
   w: \mathbb{R}^3 \rightarrow \mathbb{R}^2, \quad w(x, y, z) = \begin{pmatrix}
   \frac{x}{\sqrt{1+z^2}} \\
   \frac{z}{\sqrt{1+z^2}}
   \end{pmatrix}
   \]
   \[
   f: \mathbb{R}^2 \rightarrow \mathbb{R}, \quad f(x, y) = \langle v(x, y), v(x, y) \rangle
   \]
   and
   \[
   w \circ v: \mathbb{R}^2 \rightarrow \mathbb{R}^2
   \]
   Give an interpretation of this result. (Hint: interpret \( w \) as a bijection from \( S^2 \setminus \{0, 0, -1\} \) onto \( \mathbb{R}^2 \). Then, since the Jacobian matrix of the composition is the identity, the relation between \( w \) and \( v \) is obvious.)
4. Let \( f : \mathbb{R}^2 \to \mathbb{R} \) be
\[
f(x, y) = \frac{x^2 y \sin(\sqrt{x^2 + y^2})}{(x^2 + y^2)^{3/2}}, \quad (x, y) \neq (0, 0)
\]
Then
(a) \( \lim_{(x,y) \to (0,0)} f(x, y) = 1 \)
(b) \( \lim_{(x,y) \to (0,0)} f(x, y) = y \)
(c) \( \lim_{(x,y) \to (0,0)} f(x, y) \) does not exist
(d) \( \lim_{(x,y) \to (0,0)} f(x, y) = 0 \)

5. Let \( f : \mathbb{R}^2 \to \mathbb{R} \) be
\[
f(x, y) = \begin{cases} 
\frac{y^2}{\sqrt{y^2 + x^2}} & \text{if } (x, y) \neq (0,0) \\
0 & \text{if } (x, y) = (0,0)
\end{cases}
\]
Then
(a) \( \lim_{(x,y) \to (0,0)} \frac{\partial f}{\partial x} (x, y) = 1 \)
(b) \( \lim_{(x,y) \to (0,0)} \frac{\partial f}{\partial x} (x, y) = 0 \)
(c) \( \lim_{(x,y) \to (0,0)} \frac{\partial f}{\partial x} (x, y) \) does not exist
(d) \( \lim_{(x,y) \to (0,0)} \frac{\partial f}{\partial x} (x, y) = +\infty \)

6. Let \( f : \mathbb{R}^2 \to \mathbb{R} \) be \( f(x, y) = x^3 - 2xy + y^2 \). Then the point \( p = (2/3, 2/3) \)
(a) is a local maximum of \( f \)
(b) is not a stationary point of \( f \)
(c) is a saddle point of \( f \)
(d) is a local minimum of \( f \)

7. Let \( f \in C^2(\mathbb{R}^2) \) and \( p \in \mathbb{R}^2 \). If \( \text{Hess}_f(p) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \) then
(a) \( p \) is necessarily a local maximum
(b) \( p \) is necessarily a local minimum
(c) \( p \) is necessarily a saddle point
(d) None of above

8. Let the function \( f : \mathbb{R}^3 \to \mathbb{R} \) be \( f(x, y, z) = 2x^2 y^3 z^4 + 2x^3 y^2 - 3y^2 z - 1 \) and consider \( p = (1, 1, 1) \). Since \( f(p) = 0 \) and \( \partial f/\partial x(p) \) is not zero, the equation \( f(x, y, z) = 0 \) defines in
the neighbourhood of \( (y, z) = (1, 1) \) a function \( x = g(y, z) \) which satisfies \( g(1, 1) = 1 \) and \( f(g(y, z), y, z) = 0 \) as well as:
(a) \( \frac{\partial g}{\partial x}(1, 1) = -\frac{3}{8} \)
(b) \( \frac{\partial g}{\partial x}(1, 1) = -\frac{1}{2} \)
(c) \( \frac{\partial g}{\partial x}(1, 1) = -2 \)
(d) \( \frac{\partial g}{\partial x}(1, 1) = \frac{1}{2} \)

9. Let \( D = \{(x,y) \in \mathbb{R}^2 : x > 1 \text{ and } y > -1\} \) and let the function \( f : D \to \mathbb{R} \) be \( f(x, y) = \ln(x^2 + y) \). Then a vector \( v \) in the perpendicular direction to the level curve of \( f \) passing through point \( (2, 0) \) is

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(a) \( v = (-1/4, -1)^T \)
(b) \( v = (-4, 1)^T \)
(c) \( v = (4, 1)^T \)
(d) \( v = (1, -4)^T \)

10. State if the following statements are true or false.

(a) Let \( f : \mathbb{R}^2 \to \mathbb{R} \) be such that \( f(0, 0) = 0 \). If for all \( m \in \mathbb{R} \) we have \( \lim_{x \to 0} f(x, mx) = 0 \), then \( f \) is continuous at \((0, 0)\).
(b) Let \( f : \mathbb{R}^2 \to \mathbb{R} \). If \( f \in C^2(\mathbb{R}^2) \), then for all points \( p \in \mathbb{R}^2 \) we have
\[
\frac{\partial^2 f}{\partial x \partial y}(p) = \frac{\partial^2 f}{\partial y \partial x}(p)
\]
(c) Let \( f : \mathbb{R}^2 \to \mathbb{R} \) such that \( f \in C^2(\mathbb{R}^2) \) and let a point \( p \in \mathbb{R}^2 \). If \( p \) is a stationary point of \( f \) and if determinant of the Hessian matrix \( H_f(p) \) is strictly positive, then \( f \) admits a minimum at \( p \).
(d) Let \( f : \mathbb{R}^2 \to \mathbb{R} \). If \( f \in C^2(\mathbb{R}^2) \), then
\[
\frac{\partial f}{\partial x}(x, y) = \lim_{(h,k) \to (0,0)} \frac{f(x + h, y + k) - f(x, y)}{\sqrt{h^2 + k^2}}
\]
(e) Let \( f : \mathbb{R}^2 \to \mathbb{R} \) be a function and \( p \in \mathbb{R}^2 \). Then \( f \) is differentiable at \( p \) if and only if \( \partial f/\partial x \) and \( \partial f/\partial y \) exist at \( p \).
(f) if \( f : \mathbb{R}^2 \to \mathbb{R} \) be a function. If \( f \) is differentiable at all points of \( \mathbb{R}^2 \), then \( f \) is of class \( C^1(\mathbb{R}^2) \)
(g) Let \( f : \mathbb{R}^3 \to \mathbb{R} \), be a function that is differentiable at a point \( p \in \mathbb{R}^3 \). Then the vector
\[
v = (-\frac{\partial f}{\partial x}(p), -\frac{\partial f}{\partial y}(p), -\frac{\partial f}{\partial z}(p), 1)
\]
is perpendicular to the tangent hyperplane to the graph of \( f \) at the point \((p, f(p))\).