1. Divergence, Curl and Laplacian

(a) Let \( f : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) be defined as \( f(x, y, z) = (y + x^2, z, x^2) \). Compute \( \nabla \cdot f \), \( \nabla (\nabla \cdot f) \) and \( \nabla \times f \).

(b) Let \( f : \mathbb{R}^3 \rightarrow \mathbb{R} \) be \( f(x, y, z) = x^3 + y^2 + z \). Find \( \Delta f + \nabla \cdot (\nabla \times (\nabla f)) \).

2. Let \( f(x, y, z) = (3xyz^2, 2xy^3, -x^2yz) \) and \( \phi(x, y, z) = 3x^2 - yz \). Find \( \nabla \cdot f \), \( \nabla \times f \), \( f \cdot \nabla \phi \), \( \nabla \cdot (\nabla \phi) \) and \( \nabla \cdot (\phi f) \) at point \((1, -1, 1)\).

3. Transport equation (aka. convection-diffusion equation) is used in physics and engineering to describe phenomena where particles, energy, or other physical quantities are transferred inside a physical system due to two processes: diffusion (spreading) and convection (movement). The general transport equation is

\[
\frac{\partial c}{\partial t} + \nabla \cdot (c \vec{v}) = \nabla \cdot (D \nabla c) + S
\]

where \( \vec{v} = (v_1, v_2, v_3) \) and \( c, D, S, v_1, v_2 \) and \( v_3 \) are all real functions of \( t, x, y \) and \( z \).

(a) Write the equation without using the gradient operator \( \nabla \) and the divergence operator \( \nabla \cdot \).

(b) Verify that \( c(x, y, z) = x^2 + y^2 + z \) satisfy the steady state equation, i.e. when all derivatives with respect to time \( t \) is zero, when \( D = 5 \), \( v = (1, 2, 2) \) and \( S = 2x + 4y - 18 \).

4. Verify that

\[
\nabla \times \nabla \times f = \nabla (\nabla \cdot f) - \Delta f
\]

where \( f(x, y, z) = (f_1, f_2, f_3) \) and \( \Delta f = (\Delta f_1, \Delta f_2, \Delta f_3) \).

5. Study the change of variable given by

\[
x = \sin s \cosh t, \quad y = \cos s \sinh t.
\]

Give the Jacobian matrix, denoted \( J_\psi \), and calculate \( J_\psi^T J_\psi \). Let \( f(x, y) = f(\sin s \cosh t, \cos s \sinh t) \) a function of class \( C^2 \). Calculate

\[
\frac{\partial^2 f(x, y)}{\partial s^2} + \frac{\partial^2 f(x, y)}{\partial t^2}
\]

Use this result to give the Laplacian of a function \( f(x, y) \) of class \( C^2 \) in terms of coordinates \((s, t)\).

6. Change of coordinates between spherical and Cartesian coordinates in \( \mathbb{R}^3 \). On \( U := \{(r, \theta, \phi) : r > 0, 0 < \theta < \pi, 0 < \phi < 2\pi\} \) we consider the map

\[
x = v_1(r, \theta, \phi) = r \sin \theta \cos \phi \\
y = v_2(r, \theta, \phi) = r \sin \theta \sin \phi \\
z = v_3(r, \theta, \phi) = r \cos \theta
\]
Show that $v$ is locally invertible. Then show that for $(x,y,z) \in W := \{(x,y,z) : x > 0, y > 0, z > 0\}$ the reciprocal map $w = v^{-1}$ is given by
\[
\begin{align*}
    r &= w_1(x,y,z) = \sqrt{x^2 + y^2 + z^2} \\
    \theta &= w_2(x,y,z) = \arccos \frac{z}{\sqrt{x^2 + y^2 + z^2}} \\
    \phi &= w_3(x,y,z) = \arcsin \frac{y}{\sqrt{x^2 + y^2}}
\end{align*}
\]
Calculate the Jacobian matrix and the Jacobian determinant $w$. Give the set $w(W)$.

7. **Change of coordinates between spherical and Cartesian coordinates in $\mathbb{R}^3$.** On $U := \{(r,\theta,\phi) : r > 0, 0 < \theta < \pi, 0 < \phi < 2\pi\}$ we consider the map
\[
\begin{align*}
    x &= v_1(r,\theta,\phi) = r \sin \theta \cos \phi \\
    y &= v_2(r,\theta,\phi) = r \sin \theta \sin \phi \\
    z &= v_3(r,\theta,\phi) = r \cos \theta
\end{align*}
\]
Let $g(r,\theta,\phi)$ be a function of class $C^2(U)$. Using the previous exercise, calculate
\[
||\nabla_{x,y,z} g(r,\theta,\phi)||^2.
\]
Show that
\[
\Delta_{x,y,z} g(r,\theta,\phi) = \left[ \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \left( \frac{\partial^2}{\partial \theta^2} + \cot \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right) \right] g(r,\theta,\phi).
\]

8. For each of the following, compute $J_f$, $J_g$ and $J_{f \circ g}$.

(a) 
\[
\begin{align*}
    f(x,y) &= \left( \begin{array}{c}
        \sin x \\
        x - y
    \end{array} \right), \quad g(x,y) = \left( \begin{array}{c}
        x + y \\
        xy
    \end{array} \right)
\end{align*}
\]
(b) 
\[
\begin{align*}
    f(x,y) &= \left( \begin{array}{c}
        x + y \\
        x^3 + 2xy
    \end{array} \right), \quad g(x) = \left( \begin{array}{c}
        x
    \end{array} \right)
\end{align*}
\]

9. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ and $g : \mathbb{R}^2 \to \mathbb{R}^2$ be defined as
\[
\begin{align*}
    f(x,y) &= \left( \begin{array}{c}
        \frac{x^2 + y^2}{2}
    \end{array} \right), \quad g(x,y) = \left( \begin{array}{c}
        \sqrt{x+y} \\
        \sqrt{x^2 - y}
    \end{array} \right)
\end{align*}
\]
Compute $J_f$, $J_g$ and $J_{f \circ g}$. Is $g$ the inverse function of $f$?